The Generalized Relative Gol'dberg Order and Type: Some Remarks on Functions of Complex Variables

Tanmay Biswas
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PREFACE

The main object of this book is to discuss the generalized comparative growth analysis of entire functions of n-complex variables, which covers the important branch of complex analysis, especially the theory of analytic functions of several variables. Our book contains eight chapters.

Chapter 1 contains the introductory parts and some preliminary definitions. In chapter 2, we have developed some results related to generalized Gol’dberg order (α, β) and generalized Gol’dberg type (α, β) of entire functions of several complex variables. In chapter 3, we have proved some results about generalized relative Gol’dbergorder (α, β) of entire functions of several complex variables. In chapter 4, some inequalities using generalized relative Gol’dberg order (α, β) and generalized relative Gol’dberg lower order (α, β) of entire functions of several complex variables are established. In chapter 5, we have improved some relation connecting to generalized relative Gol’dberg type (α, β) and generalized relative Gol’dberg weak type (α, β) of entire functions of several complex variables. In chapter 6, we have derived some inequalities using generalized relative Gol’dberg type (α, β) and generalized relative Gol’dberg weak type (α, β) of entire functions of several complex variables. In chapter 7, we have discussed generalized relative Gol’dberg order (α, β) and generalized relative Gol’dberg type (α, β) based growth measure of entire functions of several complex variables. And finally, in chapter 8, we mainly focus on sum and product theorems depending on the generalized relative Gol’dberg order (α, β) and generalized relative Gol’dberg type (α, β).

To improve our results, we took help from many publications of different authors and we are thankful to them and cited their publications in the bibliography. We think this book will be very helpful for research scholars and students. We are also thankful to the Bentham Science publishers to give us the opportunity to publish this monograph.

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Chapter 1

Introduction, definitions and notations

Abstract: In this chapter, we discussed about the introductory parts connected to the entire functions of n complex variables. In this connection, we add some preliminary definitions related to different Gol'dberg growth indicators such as Gol'dberg order, Gol'dberg type etc.
Keywords: Entire functions, several complex variables, different growth indicators.
Mathematics Subject Classification (2010) : 32A15.

1.1 Introduction, definitions and notations.

The present chapter consists of some preliminary definitions in connection to the entire function $f(z)$ of n complex variables. Let $\mathbb{C}^n$ and R^n respectively denote the complex and real n-space. Also let us indicate the point $(z_1, z_2, \cdots, z_n)$, $(m_1, m_2, \cdots, m_n)$ of $\mathbb{C}^n$ or I^n by their corresponding unsuffixed symbols z, m respectively where I denotes the set of non-negative integers. The modulus of z, denoted by $|z|$, is defined as $|z| = (|z_1|^2 + \cdots + |z_n|^2)^{\frac{1}{2}}$. If the coordinates of the vector m are non-negative integers, then z^m will denote $z_1^{m_1} \cdots z_n^{m_n}$ and $\|m\| = m_1 + \cdots + m_n$.

If $D \subseteq \mathbb{C}^n$ ($\mathbb{C}^n$ denote the n-dimensional complex space) be an arbitrary bounded complex n-circular domain with center at the origin of coordinates then for any entire function $f(z)$ of n complex variables and $R > 0$, $M_{f,D}(R)$ may be define as $M_{f,D}(R) = \sup_{z \in D_R} |f(z)|$ where a point $z \in D_R$ if and only if $\frac{z}{R} \in D$. If $f(z)$ is non-constant, then $M_{f,D}(R)$ is strictly increasing and its inverse $M_{f,D}^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists such that $\lim_{R \to \infty} M_{f,D}^{-1}(R) = \infty$.

Considering this, the Gol'dberg order and Gol'dberg lower order (cf. [1, 2]) of an entire function $f(z)$ with respect to any bounded complete n-circular domain D with center at all the origin $\mathbb{C}^n$ are given by

$$\begin{matrix} \rho_D(f) \\ \lambda_D(f) \end{matrix} = \lim_{R \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \log M_{f,D}(R)}{\log R}.$$

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It is well known that $\rho_D(f)$ is independent of the choice of the domain D, and therefore we write $\rho(f)$ instead of $\rho_D(f)$ (respectively $\lambda(f)$ instead of $\lambda_D(f)$) (cf. [1, 2]).

For any bounded complete n-circular domain D, an entire function of n complex variables for which Gol'dberg order and Gol'dberg lower order are the same is said to be of regular growth. Functions which are not of regular growth are said to be of irregular growth.

To compare the relative growth of two entire functions of n complex variables having same non-zero finite Gol'dberg order, one may introduce the definition of Gol'dberg type and Gol'dberg lower type in the following manner:

Definition 1.1.1 *(cf. [1, 2]) The Gol'dberg type and Gol'dberg lower type respectively denoted by $\sigma_D(f)$ and $\overline{\sigma}_D(f)$ of an entire function $f(z)$ of n complex variables with respect to any bounded complete n-circular domain D with center at all the origin $\mathbb{C}^n$ are defined as follows:*

$$\begin{matrix} \sigma_D(f) \\ \overline{\sigma}_D(f) \end{matrix} = \lim_{R\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log M_{f,D}(R)}{(R)^{\rho(f)}},\ 0 < \rho(f) < \infty.$$

Analogously to determine the relative growth of two entire functions of n complex variables having same non-zero finite Gol'dberg lower order, one may introduce the definition of Gol'dberg weak type in the following way:

Definition 1.1.2 *The Gol'dberg weak type denoted by $\tau_D(f)$ of an entire function $f(z)$ of n complex variables with respect to any bounded complete n-circular domain D with center at all the origin $\mathbb{C}^n$ is defined as follows:*

$$\tau_D(f) = \liminf_{R\to\infty} \frac{\log M_{f,D}(R)}{(R)^{\lambda(f)}},\ 0 < \lambda(f) < \infty.$$

Also one may define the Gol'dberg upper weak type denoted by $\overline{\tau}_D(f)$ in the following manner :

$$\overline{\tau}_D(f) = \limsup_{R\to\infty} \frac{\log M_{f,D}(R)}{(R)^{\lambda(f)}},\ 0 < \lambda(f) < \infty.$$

Gol'dberg has shown that [2] Gol'dberg type depends on the domain D. Hence all the growth indicators define in Definition 1.1.1 and Definition 1.1.2 are also depend on D.

In the sequel the following two notations are used:

$$\log^{[k]} R = \log(\log^{[k-1]} R) \text{ for } k = 1, 2, 3, \cdots ;$$
$$\log^{[0]} R = R$$

and

$$\exp^{[k]} R = \exp(\exp^{[k-1]} R) \text{ for } k = 1, 2, 3, \cdots ;$$
$$\exp^{[0]} R = R.$$

Taking this into account the, one can give the definitions of generalized Gol'dberg order $\rho_D^{(l)}(f)$ and generalized Gol'dberg lower order $\lambda_D^{(l)}(f)$ in the following way:

Definition 1.1.3 *The generalized Gol'dberg order* $\rho_D^{(l)}(f)$ *and generalized Gol'dberg lower order* $\lambda_D^{(l)}(f)$ *of an entire function* $f(z)$ *of* n *complex variables with respect to any bounded complete* n*-circular domain* D *with center at all the origin* $\mathbb{C}^n$ *are defined as follows:*

$$\begin{matrix} \rho_D^{(l)}(f) \\ \lambda_D^{(l)}(f) \end{matrix} = \lim_{R\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[l]} M_{f,D}(R)}{\log R},$$

where l *is any positive integer such that* $l \geq 2$.

In the line of Gol'dberg (cf. [1, 2]), one can easily verify that $\rho_D^{(l)}(f)$ and $\lambda_D^{(l)}(f)$ are independent of the choice of the domain D, and therefore we write $\rho^{(l)}(f)$ instead of $\rho_D^{(l)}(f)$ and $\lambda^{(l)}(f)$ instead of $\lambda_D^{(l)}(f)$.

This definition extended the Gol'dberg order $\rho(f)$ and Gol'dberg lower order $\lambda(f)$ of an entire function $f(z)$ of n complex variables with respect to any bounded complete n-circular domain D since this correspond to the particular case $\rho^{(2)}(f) = \rho(f)$ and $\lambda^{(2)}(f) = \lambda(f)$.

However, an entire function $f(z)$ for which $\rho^{(l)}(f)$ and $\lambda^{(l)}(f)$ are the same is called a function of regular generalized Gol'dberg growth. Otherwise, $f(z)$ is said to be irregular generalized Gol'dberg growth.

The following two definitions are the natural consequences of the above study:

Definition 1.1.4 *The generalized Gol'dberg type* $\sigma_f^{[l]}$ *and generalized Gol'dberg lower type* $\overline{\sigma}_f^{[l]}$ *of an entire function* $f(z)$ *of* n *complex variables with respect to any bounded complete* n*-circular domain* D *with center at all the origin* $\mathbb{C}^n$ *are defined as*

$$\begin{matrix} \sigma_D^{(l)}(f) \\ \overline{\sigma}_D^{(l)}(f) \end{matrix} = \lim_{R\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[l-1]} M_{f,D}(R)}{R^{\rho^{(l)}(f)}}, \ 0 < \rho^{(l)}(f) < \infty,$$

where l *is any positive integer such that* $l \geq 2$. *Moreover, when* $l = 2$ *then* $\sigma_D^{(2)}(f)$ *and* $\overline{\sigma}_D^{(2)}(f)$ *are correspondingly denoted as* $\sigma_D(f)$ *and* $\overline{\sigma}_D(f)$.

Similarly, extending the notion of Gol'dberg weak type, one can define generalized Gol'dberg weak type in the following manner:

Definition 1.1.5 *The generalized Gol'dberg weak type* $\tau_D^{(l)}(f)$ *for any positive integer* $l \geq 2$ *of an entire function* $f(z)$ *of* n *complex variables with respect to any bounded complete* n*-circular domain* D *with center at all the origin* $\mathbb{C}^n$ *having finite positive generalized Gol'dberg lower order* $\lambda^{(l)}(f)$ *are defined by*

$$\tau_D^{(l)}(f) = \liminf_{R\to\infty} \frac{\log^{[l-1]} M_{f,D}(R)}{R^{\lambda^{(l)}(f)}}, \ 0 < \lambda^{(l)}(f) < \infty.$$

Also one may define the generalized Gol'dberg upper weak type denoted by $\overline{\tau}_D^{(l)}(f)$ *in the following way:*

$$\overline{\tau}_D^{(l)}(f) = \limsup_{R\to\infty} \frac{\log^{[l-1]} M_{f,D}(R)}{R^{\lambda^{(l)}(f)}}, \ 0 < \lambda^{(l)}(f) < \infty.$$

Chapter 2

Generalized Gol'dberg order (α, β) and generalized Gol'dberg type (α, β) of entire functions of several complex variables

Abstract: In this chapter, first we introduce the definitions of generalized Gol'dberg order (α, β), generalized hyper Gol'dberg order (α, β) generalized logarithmic Gol'dberg order (α, β), generalized Gol'dberg type (α, β) and generalized Gol'dberg weak type (α, β) of entire functions of several complex variables and then using these growth indicators, we discuss of some related growth properties of entire functions of n complex variables, where α, β are continuous non-negative functions defined on $(-\infty, +\infty)$.
Keywords: Increasing function, generalized Gol'dberg order (α, β), generalized hyper Gol'dberg order (α, β), generalized logarithmic Gol'dberg order (α, β), generalized Gol'dberg type (α, β), generalized Gol'dberg weak type (α, β).
Mathematics Subject Classification (2010) : 32A15.

2.1 Introduction.

The Gol'dberg order and Gol'dberg type of an entire function $f(z)$ with respect to any bounded complete n-circular domain D with center at all the origin $\mathbb{C}^n$ which are generally used in computational purpose are classical. Datta et al. [1] defined the concept of (p, q)-th Gol'dberg order of an entire function $f(z)$ for any bounded complete n-circular domain D with center at all the origin $\mathbb{C}^n$ where p and q are any positive integers with $p \geq q \geq 1$. Extending this notion, here in this chapter we wish to introduce the definitions of generalized Gol'dberg order (α, β) and generalized Gol'dberg type (α, β) of an entire functions of several complex variables and establish some related growth properties of entire functions of several complex variables.

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2.2 Preliminary remarks and definitions.

Throughout the book we assume L be a class of continuous non-negative functions α defined on $(-\infty, +\infty)$ such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow +\infty$ as $x \to +\infty$. For any $\alpha \in L$, we say that $\alpha \in L^0$, if $\alpha(cx) = (1 + o(1))\,\alpha(x)$ as $x_0 \leq x \to +\infty$ for each $c \in (0, +\infty)$. Clearly, $L^0 \subset L$.

Further we assume that throughout the book, unless specified later, $\alpha, \alpha_1, \alpha_2, \gamma, \beta, \beta_1$ and β_2 always denote the functions belonging to L^0. Now considering this, we introduce the definition of the generalized Gol'dberg order (α, β) and generalized Gol'dberg lower order (α, β) of an entire function $f(z)$ with respect to any bounded complete n-circular domain D with center at all the origin $\mathbb{C}^n$ which are as follows:

Definition 2.2.1 *The generalized Gol'dberg order (α, β) and generalized Gol'dberg lower order (α, β) of an entire function $f(z)$ with respect to any bounded complete n-circular domain D with center at all the origin $\mathbb{C}^n$ are defined as:*

$$\begin{matrix} \rho_D^{(\alpha,\beta)}[f] \\ \lambda_D^{(\alpha,\beta)}[f] \end{matrix} = \lim_{R\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(M_{f,D}(R))}{\beta(R)}.$$

Definition of (p, q)-th Gol'dberg order is a special case of Definition 2.2.1 for $\alpha(R) = \log^{[p]} R$ and $\beta(R) = \log^{[q]} R$.

The function $f(z)$ is said to be of regular generalized Gol'dberg (α, β) growth when generalized Gol'dberg order (α, β) and generalized Gol'dberg lower order (α, β) of $f(z)$ are the same. Functions which are not of regular generalized Gol'dberg (α, β) growth are said to be of irregular generalized Gol'dberg (α, β) growth.

Now in order to refine the growth scale namely the generalized Gol'dberg order (α, β), we introduce the definitions of another growth indicators, called generalized Gol'dberg type (α, β) and generalized Gol'dberg lower type (α, β) respectively of an entire function $f(z)$ with respect to any bounded complete n-circular domain D with center at all the origin $\mathbb{C}^n$ which are as follows:

Definition 2.2.2 *The generalized Gol'dberg type (α, β) and generalized Gol'dberg lower type (α, β) of an entire function $f(z)$ with respect to any bounded complete n-circular domain D with center at all the origin $\mathbb{C}^n$ having finite positive generalized Gol'dberg order (α, β) $\left(0 < \rho_D^{(\alpha,\beta)}[f] < \infty\right)$ are defined as :*

$$\begin{matrix} \sigma_D^{(\alpha,\beta)}[f] \\ \overline{\sigma}_D^{(\alpha,\beta)}[f] \end{matrix} = \lim_{R\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\exp(\alpha(M_{f,D}(R)))}{(\exp(\beta(r)))^{\rho_D^{(\alpha,\beta)}[f]}}.$$

It is obvious that $0 \leq \overline{\sigma}_D^{(\alpha,\beta)}[f] \leq \sigma_D^{(\alpha,\beta)}[f] \leq \infty$.

Analogously to determine the relative growth of two entire functions of n complex variables having same non-zero finite generalized Gol'dberg lower order (α, β), one may introduce the definition of generalized Gol'dberg weak type (α, β) and generalized

Gol'dberg upper weak type (α,β) of an entire function $f(z)$ with respect to any bounded complete n-circular domain D with center at all the origin $\mathbb{C}^n$ having finite positive generalized Gol'dberg lower order (α,β), $\lambda_D^{(\alpha,\beta)}[f]$ in the following way:

Definition 2.2.3 *The generalized Gol'dberg upper weak type (α,β) denoted by $\overline{\tau}_D^{(\alpha,\beta)}[f]$ and generalized Gol'dberg weak type (α,β) denoted by $\tau_D^{(\alpha,\beta)}[f]$ of an entire function $f(z)$ with respect to any bounded complete n-circular domain D with center at all the origin $\mathbb{C}^n$ having finite positive generalized Gol'dberg lower order (α,β) $\left(0<\lambda_D^{(\alpha,\beta)}[f]<\infty\right)$ are defined as :*

$$\begin{matrix}\overline{\tau}_D^{(\alpha,\beta)}[f] \\ \tau_D^{(\alpha,\beta)}[f]\end{matrix} = \lim_{R\to+\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\exp(\alpha(M_{f,D}(R)))}{(\exp(\beta(r)))^{\lambda_D^{(\alpha,\beta)}[f]}}.$$

It is obvious that $0\leq\tau_D^{(\alpha,\beta)}[f]\leq\overline{\tau}_D^{(\alpha,\beta)}[f]\leq\infty$.

Remark 2.2.1 *As Gol'dberg has shown that (see [2]) Gol'dberg type depends on the domain D, so in general all the growth indicators defined in Definition 2.2.2 and Definition 2.2.3 also depend on D.*

Now one may give the following definitions of generalized hyper Gol'dberg order (α,β) and generalized logarithmic Gol'dberg order (α,β) of an entire function $f(z)$ with respect to any bounded complete n-circular domain D with center at all the origin $\mathbb{C}^n$ in the following way:

Definition 2.2.4 *The generalized hyper Gol'dberg order (α,β) and generalized hyper Gol'dberg lower order (α,β) of an entire function $f(z)$ with respect to any bounded complete n-circular domain D with center at all the origin $\mathbb{C}^n$ are defined as:*

$$\begin{matrix}\overline{\rho}_D^{(\alpha,\beta)}[f] \\ \overline{\lambda}_D^{(\alpha,\beta)}[f]\end{matrix} = \lim_{R\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\alpha(\log(M_{f,D}(R)))}{\beta(R)}.$$

Definition 2.2.5 *The generalized logarithmic Gol'dberg order (α,β) and generalized logarithmic Gol'dberg lower order (α,β) of an entire function $f(z)$ with respect to any bounded complete n-circular domain D with center at all the origin $\mathbb{C}^n$ are defined as:*

$$\begin{matrix}\underline{\rho}_D^{(\alpha,\beta)}[f] \\ \underline{\lambda}_D^{(\alpha,\beta)}[f]\end{matrix} = \lim_{R\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\alpha(M_{f,D}(R))}{\beta(\log R)}.$$

2.3 Main Results.

In this section we state the main results of this chapter.

Theorem 2.3.1 *Let $f(z)$ be any entire function of n complex variables. Then generalized Gol'dberg order (α,β) and generalized Gol'dberg lower order (α,β) of $f(z)$ are independent of the choice of the domain D.*

Chapter 3

Generalized relative Gol'dberg order (α, β) of entire functions of several complex variables

Abstract: The aim of the chapter is to introduce the concepts of generalized relative Gol'dberg order (α, β), generalized relative hyper Gol'dberg order (α, β), and generalized relative logarithmic Gol'dberg order (α, β) of an entire function of several complex variables with respect to another entire function of several complex variables, where α, β are continuous non-negative functions defined on $(-\infty, +\infty)$. Then we discuss some growth analysis of entire functions of several complex variables. Also we established some integral representations of the above growth indicators.
Keywords: Entire functions of several complex variables, increasing function, Generalized relative Gol'dberg order (α, β), generalized relative hyper Gol'dberg order (α, β), generalized relative logarithmic Gol'dberg order (α, β), generalized relative logarithmic Gol'dberg lower order (α, β).
Mathematics Subject Classification (2010) : 32A15.

3.1 Introduction.

The Gol'dberg order and Gol'dberg type of an entire function $f(z)$ with respect to any bounded complete n-circular domain D with center at all the origin $\mathbb{C}^n$ which are generally used in computational purpose are classical. Mondal et al. [1] defined the concept of relative Gol'dberg order between two entire functions $f(z)$ and $g(z)$ for any bounded complete n-circular domain D with center at all the origin $\mathbb{C}^n$. Extending this notion, here in this chapter we wish to introduce the definition of generalized relative Gol'dberg order (α, β) and generalized relative Gol'dberg lower order (α, β) between two entire functions of several complex variables and establish some related growth properties with their integral representations.

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3.2 Preliminary remarks and definitions.

First we introduce the definitions of the generalized relative Gol'dberg order (α, β) and generalized relative Gol'dberg lower order (α, β) of an entire function in $\mathbb{C}^n$ with respect to another entire function of several variables in the following way:

Definition 3.2.1 *Let $f(z)$ and $g(z)$ be any two entire functions of n complex variables. The generalized relative Gol'dberg order (α, β) of $f(z)$ with respect to $g(z)$ is defined by:*

$$\rho_D^{(\alpha,\beta)}[f]_g = \limsup_{R\to\infty} \frac{\alpha(M_{g,D}^{-1}(M_{f,D}(R)))}{\beta(R)}.$$

Definition 3.2.2 *Let $f(z)$ and $g(z)$ be any two entire functions of n complex variables. The growth indicator $\rho_D^{(\alpha,\beta)}[f]_g$ is alternatively defined as : The integral*

$$\int_{R_0}^{\infty} \frac{\exp(\alpha(M_{g,D}^{-1}(M_{f,D}(R))))}{(\exp\beta(R))^{k+1}} dR \,(R_0 > 0)$$

converges for $k > \rho_D^{(\alpha,\beta)}[f]_g$ and diverges for $k < \rho_D^{(\alpha,\beta)}[f]_g$.

Definition 3.2.3 *Let $f(z)$ and $g(z)$ be any two entire functions of n complex variables. The generalized relative Gol'dberg lower order (α, β) of $f(z)$ with respect to $g(z)$ is defined as:*

$$\lambda_D^{(\alpha,\beta)}[f]_g = \liminf_{R\to\infty} \frac{\alpha(M_{g,D}^{-1}(M_{f,D}(R)))}{\beta(R)}.$$

Definition 3.2.4 *Let $f(z)$ and $g(z)$ be any two entire functions of n complex variables. The growth indicator $\lambda_D^{(\alpha,\beta)}[f]_g$ is alternatively defined as : The integral*

$$\int_{R_0}^{\infty} \frac{\exp(\alpha(M_{g,D}^{-1}(M_{f,D}(R))))}{(\exp\beta(R))^{k+1}} dR \,(R_0 > 0)$$

converges for $k > \lambda_D^{(\alpha,\beta)}[f]_g$ and diverges for $k < \lambda_D^{(\alpha,\beta)}[f]_g$.

An entire function $f(z)$ of n complex variables for which $\rho_D^{(\alpha,\beta)}[f]_g$ and $\lambda_D^{(\alpha,\beta)}[f]_g$ are the same is called a function of regular generalized relative Gol'dberg (α, β) growth with respect to an entire function $g(z)$ of n complex variables. Otherwise, $f(z)$ is said to be irregular generalized relative Gol'dberg (α, β) growth with respect to $g(z)$.

Now a question may arise about the equivalence of the definitions of generalized relative Gol'dberg order (α, β) and generalized relative Gol'dberg lower order (α, β) with their integral representations. In the next section we would like to establish such equivalence of Definition 3.2.1 and Definition 3.2.3 with Definition 3.2.2 and Definition 3.2.4 respectively and also investigate some growth properties related to generalized relative Gol'dberg order (α, β) and generalized relative Gol'dberg lower order (α, β) of an entire functions of n complex variables with respect to another entire function of n complex variables.

3.3 Lemma.

In this section we present a lemma which will be needed in the sequel.

Lemma 3.3.1 *Let the integral* $\int\limits_{R_0}^{\infty} \frac{\exp(\alpha(M_{g,D}^{-1}(M_{f,D}(R))))}{(\exp\beta(R))^{k+1}} dR$ $(R_0 > 0)$ *converges for* $0 < k < \infty$. *Then*

$$\lim_{R\to\infty} \frac{\exp(\alpha(M_{g,D}^{-1}(M_{f,D}(R))))}{(\exp\beta(R))^{k}} = 0.$$

Proof. Since the integral $\int\limits_{R_0}^{\infty} \frac{\exp(\alpha(M_{g,D}^{-1}(M_{f,D}(R))))}{(\exp\beta(R))^{k+1}} dR$ is convergent for $0 < k < \infty$, given ε (> 0) there exists a number $\Re = \Re(\varepsilon)$ such that

$$\int\limits_{R_0}^{\infty} \frac{\exp(\alpha(M_{g,D}^{-1}(M_{f,D}(R))))}{(\exp\beta(R))^{k+1}} dR < \varepsilon \text{ for } R_0 > \Re.$$

i.e., for $R_0 > \Re$,

$$\int\limits_{R_0}^{R_0+R} \frac{\exp(\alpha(M_{g,D}^{-1}(M_{f,D}(R))))}{(\exp\beta(R))^{k+1}} dR < \varepsilon.$$

Since $\exp(\alpha(M_{g,D}^{-1}(M_{f,D}(R))))$ is an increasing function of R, so

$$\int\limits_{R_0}^{R_0+R} \frac{\exp(\alpha(M_{g,D}^{-1}(M_{f,D}(R))))}{(\exp\beta(R))^{k+1}} dR \geq \frac{\exp(\alpha(M_{g,D}^{-1}(M_{f,D}(R))))}{(\exp\beta(R_0))^{k+1}} \cdot (\exp\beta(R_0)$$

i.e., for all large values of R,

$$\int\limits_{R_0}^{R_0+R} \frac{\exp(\alpha(M_{g,D}^{-1}(M_{f,D}(R))))}{(\exp\beta(R))^{k+1}} dR \geq \frac{\exp(\alpha(M_{g,D}^{-1}(M_{f,D}(R))))}{(\exp\beta(R_0))^{k}}$$

$$\textit{i.e.,}\ \frac{\exp(\alpha(M_{g,D}^{-1}(M_{f,D}(R))))}{(\exp\beta(R_0))^{k}} < \varepsilon \text{ for } R_0 > \Re,$$

from which it follows that

$$\lim_{R\to\infty} \frac{\exp(\alpha(M_{g,D}^{-1}(M_{f,D}(R))))}{(\exp\beta(R))^{k}} = 0.$$

This proves the lemma. ■

Chapter 4

Some inequalities using generalized relative Gol'dberg order (α, β) and generalized relative Gol'dberg lower order (α, β) of entire functions of several complex variables

Abstract: In this chapter, Some inequalities using generalized Gol'dberg order (α, β), generalized Gol'dberg lower order (α, β), generalized relative Gol'dberg order (α, β) and generalized relative Gol'dberg lower order (α, β) of entire functions of several complex variables are established, where α, β are continuous non-negative functions defined on $(-\infty, +\infty)$.
Keywords: Entire function, several complex variables, generalized Gol'dberg order (α, β), generalized relative Gol'dberg order (α, β), increasing function.
Mathematics Subject Classification (2010) : 32A15.

4.1 Introduction.

The relative Gol'dberg order of an entire function of n complex variables gives a quantitative assessment of how different functions scale each other and until what extent they are self-similar in growth. In Chapter Two and Chapter Three, we give relevant notations and definitions of $\rho^{(\alpha,\beta)}[f]$, $\lambda^{(\alpha,\beta)}[f]$, $\rho^{(\alpha,\beta)}[f]_g$, $\lambda^{(\alpha,\beta)}[f]_g$ etc. In this chapter we discuss some growth rates of entire functions of n complex variables on the basis of the generalized Gol'dberg order (α, β), generalized Gol'dberg lower order (α, β), generalized relative Gol'dberg order (α, β) and generalized relative Gol'dberg lower order (α, β) where $\alpha, \beta \in L_0$. Further we assume that throughout the present chapter α, β and γ always denote the functions belonging to L^0.

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4.2 Main Results.

In this section we present the main results of this chapter.

Theorem 4.2.1 *Let* $f(z)$ *and* $g(z)$ *be two entire functions of* n *complex variables such that* $0 < \lambda^{(\gamma,\beta)}[f] \leq \rho^{(\gamma,\beta)}[f] < \infty$ *and* $0 < \lambda^{(\gamma,\alpha)}[g] \leq \rho^{(\gamma,\alpha)}[g] < \infty$ *. Then*

$$\frac{\lambda^{(\gamma,\beta)}[f]}{\rho^{(\gamma,\alpha)}[g]} \leq \lambda^{(\alpha,\beta)}[f]_g \leq \min\left\{\frac{\lambda^{(\gamma,\beta)}[f]}{\lambda^{(\gamma,\alpha)}[g]}, \frac{\rho^{(\gamma,\beta)}[f]}{\rho^{(\gamma,\alpha)}[g]}\right\}$$
$$\leq \max\left\{\frac{\lambda^{(\gamma,\beta)}[f]}{\lambda^{(\gamma,\alpha)}[g]}, \frac{\rho^{(\gamma,\beta)}[f]}{\rho^{(\gamma,\alpha)}[g]}\right\} \leq \rho^{(\alpha,\beta)}[f]_g \leq \frac{\rho^{(\gamma,\beta)}[f]}{\lambda^{(\gamma,\alpha)}[g]}.$$

Proof. From the definitions of $\rho^{(\gamma,\beta)}[f]$ and $\lambda^{(\gamma,\beta)}[f]$, we have for all sufficiently large values of R that

$$M_{f,D}(R) \leq \gamma^{-1}((\rho^{(\gamma,\beta)}[f] + \varepsilon)\,\beta(R)), \tag{49}$$
$$M_{f,D}(R) \geq \gamma^{-1}((\lambda^{(\gamma,\beta)}[f] - \varepsilon)\,\beta(R)) \tag{50}$$

and also for a sequence of values of R tending to infinity we get that

$$M_{f,D}(R) \geq \gamma^{-1}((\rho^{(\gamma,\beta)}[f] - \varepsilon)\,\beta(R)), \tag{51}$$
$$M_{f,D}(R) \leq \gamma^{-1}((\lambda^{(\gamma,\beta)}[f] + \varepsilon)\,\beta(R)). \tag{52}$$

Similarly from the definitions of $\rho^{(\gamma,\alpha)}[g]$ and $\lambda^{(\gamma,\alpha)}[g]$, it follows for all sufficiently large values of R that

$$M_{g,D}(R) \leq \gamma^{-1}((\rho^{(\gamma,\alpha)}[g] + \varepsilon)\,\alpha(R))$$
$$i.e.,\ R \leq M_{g,D}^{-1}\left(\gamma^{-1}((\rho^{(\gamma,\alpha)}[g] + \varepsilon)\,\alpha(R))\right)$$
$$i.e.,\ M_{g,D}^{-1}(R) \geq \alpha^{-1}\left(\frac{\gamma(R)}{(\rho^{(\gamma,\alpha)}[g] + \varepsilon)}\right), \tag{53}$$

$$M_{g,D}(R) \geq \gamma^{-1}((\lambda^{(\gamma,\alpha)}[g] - \varepsilon)\,\alpha(R))$$
$$i.e.,\ M_{g,D}^{-1}(R) \leq \alpha^{-1}\left(\frac{\gamma(R)}{(\lambda^{(\gamma,\alpha)}[g] - \varepsilon)}\right) \tag{54}$$

and for a sequence of values of R tending to infinity we obtain that

$$M_{g,D}(R) \geq \gamma^{-1}((\rho^{(\gamma,\alpha)}[g] - \varepsilon)\,\alpha(R))$$
$$i.e.\ M_{g,D}^{-1}(R) \leq \alpha^{-1}\left(\frac{\gamma(R)}{(\rho^{(\gamma,\alpha)}[g] - \varepsilon)}\right), \tag{55}$$

$$M_{g,D}(R) \leq \gamma^{-1}((\lambda^{(\gamma,\alpha)}[g] + \varepsilon)\,\alpha(R))$$
$$i.e.,\ M_{g,D}^{-1}(R) \geq \alpha^{-1}\left(\frac{\gamma(R)}{(\lambda^{(\gamma,\alpha)}[g] + \varepsilon)}\right). \tag{56}$$

Now from (51) and in view of (53), for a sequence of values of R tending to infinity we get that

$$\alpha(M_{g,D}^{-1}(M_{f,D}(R))) \geq \alpha(M_{g,D}^{-1}\left(\gamma^{-1}(\left(\rho^{(\gamma,\beta)}[f]-\varepsilon\right)\beta(R))\right))$$

$$i.e.,\ \alpha(M_{g,D}^{-1}(M_{f,D}(R))) \geq \alpha\left(\alpha^{-1}\left(\frac{\gamma(\gamma^{-1}((\rho^{(\gamma,\beta)}[f]-\varepsilon)\beta(R)))}{(\rho^{(\gamma,\alpha)}[g]+\varepsilon)}\right)\right)$$

$$= \frac{\left(\rho^{(\gamma,\beta)}[f]-\varepsilon\right)}{(\rho^{(\gamma,\alpha)}[g]+\varepsilon)}\beta(R)$$

$$i.e.,\ \frac{\alpha(M_{g,D}^{-1}(M_{f,D}(R)))}{\beta(R)} \geq \frac{\left(\rho^{(\gamma,\beta)}[f]-\varepsilon\right)}{(\rho^{(\gamma,\alpha)}[g]+\varepsilon)}.$$

As $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\rho_{(\alpha,\beta)}[f] \geq \frac{\rho^{(\gamma,\beta)}[f]}{\rho^{(\gamma,\alpha)}[g]}. \tag{57}$$

Analogously, from (50) and in view of (56) it follows for a sequence of values of R tending to infinity that

$$\alpha(M_{g,D}^{-1}(M_{f,D}(R))) \geq \alpha(M_{g,D}^{-1}\left(\gamma^{-1}(\left(\lambda^{(\gamma,\alpha)}[g]-\varepsilon\right)\beta(R))\right))$$

$$i.e.,\ \alpha(M_{g,D}^{-1}(M_{f,D}(R))) \geq \alpha\left(\alpha^{-1}\left(\frac{\gamma(\gamma^{-1}((\lambda^{(\gamma,\beta)}[f]-\varepsilon)\beta(R)))}{(\lambda^{(\gamma,\alpha)}[g]+\varepsilon)}\right)\right)$$

$$= \frac{\left(\lambda^{(\gamma,\beta)}[f]-\varepsilon\right)}{(\lambda^{(\gamma,\alpha)}[g]+\varepsilon)}\beta(R)$$

$$i.e.,\ \frac{\alpha(M_{g,D}^{-1}(M_{f,D}(R)))}{\beta(R)} \geq \frac{\left(\lambda^{(\gamma,\beta)}[f]-\varepsilon\right)}{(\lambda^{(\gamma,\alpha)}[g]+\varepsilon)}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\rho^{(\alpha,\beta)}[f]_g \geq \frac{\lambda^{(\gamma,\beta)}[f]}{\lambda^{(\gamma,\alpha)}[g]}. \tag{58}$$

Again in view of (54), we have from (49) for all sufficiently large values of R that

$$\alpha(M_{g,D}^{-1}(M_{f,D}(R))) \leq \alpha(M_{g,D}^{-1}\left(\gamma^{-1}(\left(\rho^{(\gamma,\beta)}[f]+\varepsilon\right)\beta(R))\right))$$

$$i.e.,\ \alpha(M_{g,D}^{-1}(M_{f,D}(R))) \leq \alpha\left(\alpha^{-1}\left(\frac{\gamma(\gamma^{-1}((\rho^{(\gamma,\beta)}[f]+\varepsilon)\beta(R)))}{(\lambda^{(\gamma,\alpha)}[g]-\varepsilon)}\right)\right)$$

$$= \frac{\left(\rho^{(\gamma,\beta)}[f]+\varepsilon\right)}{(\lambda^{(\gamma,\alpha)}[g]-\varepsilon)}\beta(R)$$

Chapter 5

Generalized relative Gol'dberg type (α, β) and generalized relative Gol'dberg weak type (α, β) of entire functions of several complex variables

Abstract: In this chapter, we develop some growth properties of entire functions of n complex variables relating to generalized relative Gol'dberg order (α, β), generalized relative Gol'dberg type (α, β) and generalized relative Gol'dberg weak type (α, β).We also establish integral representations of generalized relative Gol'dberg type and weak type (α, β) of entire function of several complex variables and derive some interesting results, where α, β are continuous non-negative functions defined on $(-\infty, +\infty)$.
Keywords: Generalized relative Gol'dberg order (α, β), generalized relative Gol'dberg lower order (α, β), generalized relative Gol'dberg type (α, β), generalized relative Gol'dberg weak type (α, β), increasing function.
Mathematics Subject Classification (2010) : 32A15.

5.1 Introduction.

Mondal et al. [1] defined the concept of relative Gol'dberg order between two entire functions $f(z)$ and $g(z)$ for any bounded complete n-circular domain D with center at all the origin $\mathbb{C}^n$. Extending this notion, we have already introduced the definitions of generalized relative Gol'dberg order (α, β) and generalized relative Gol'dberg lower order (α, β) between two entire functions of several complex variables Now to compare the growth of entire functions of several complex variables having the same generalized relative Gol'dberg order (α, β) or generalized relative Gol'dberg lower order (α, β), we wish to introduce the definition of generalized relative Gol'dberg type (α, β) and generalized

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relative Gol'dberg weak type (α, β) of an entire function of several complex variables with respect to another entire function of several complex variables and establish their integral representations. We also investigate their equivalence relations under certain conditions.

5.2 Preliminary remarks and definitions.

The definitions of generalized relative Gol'dberg order (α, β) and generalized relative Gol'dberg lower order (α, β) of $f(z)$ with respect to $g(z)$ where $f(z)$ and $g(z)$ be any two entire functions of n complex variables are as follows:

Definition 5.2.1 *Let $f(z)$ and $g(z)$ be any two entire functions of n complex variables. The generalized relative Gol'dberg order (α, β) of $f(z)$ with respect to $g(z)$ is defined by:*

$$\rho^{(\alpha,\beta)}[f]_g = \limsup_{R\to\infty} \frac{\alpha(M_{g,D}^{-1}(M_{f,D}(R)))}{\beta(R)}.$$

The generalized relative Gol'dberg lower order (α, β) of $f(z)$ with respect to $g(z)$ is defined as:

$$\lambda^{(\alpha,\beta)}[f]_g = \liminf_{R\to\infty} \frac{\alpha(M_{g,D}^{-1}(M_{f,D}(R)))}{\beta(R)}.$$

In order to define the above growth scale, now we intend to introduce the definition of an another growth indicator, called generalized relative Gol'dberg type (α, β) of an entire function of n complex variables with respect to another entire function of n complex variables as follows:

Definition 5.2.2 *Let $f(z)$ and $g(z)$ be any two entire functions of n complex variables.The generalized relative Gol'dberg type (α, β) of entire function $f(z)$ with respect to the entire function $g(z)$ having finite positive generalized relative Gol'dberg order (α, β) denoted by $\rho^{(\alpha,\beta)}[f]_g$ $\left(0 < \rho^{(\alpha,\beta)}[f]_g < \infty\right)$ is defined as :*

$$\sigma_D^{(\alpha,\beta)}[f]_g = \inf\left\{ \begin{array}{c} \phi > 0 : M_{f,D}(R) < M_{g,D}\left[\alpha^{-1}\log\left(\phi\,(\exp(\beta(R)))^{\rho^{(\alpha,\beta)}[f]_g}\right)\right] \\ \textit{for all } R > R_0(\phi) > 0 \end{array} \right\}$$
$$= \limsup_{R\to\infty} \frac{\exp(\alpha(M_{g,D}^{-1}(M_{f,D}(R))))}{(\exp(\beta(R)))^{\rho^{(\alpha,\beta)}[f]_g}}.$$

The above definition can alternatively defined in the following manner:

Definition 5.2.3 *Let $f(z)$ and $g(z)$ be any two entire functions of n complex variables having finite positive generalized relative Gol'dberg order (α, β) denoted by $\rho^{(\alpha,\beta)}[f]_g$ $\left(0 < \rho^{(\alpha,\beta)}[f]_g < \infty\right)$,then the generalized relative Gol'dberg type (α, β) denoted by $\sigma_D^{(\alpha,\beta)}[f]_g$ of entire function $f(z)$ with respect to the entire function $g(z)$ is define as: The integral $\int_{R_0}^{\infty} \frac{\exp^{[2]}(\alpha(M_{g,D}^{-1}(M_{f,D}(R))))}{\left[\exp\left((\exp(\beta(R)))^{\rho^{(\alpha,\beta)}[f]_g}\right)\right]^{k+1}} dR$ $(R_0 > 0)$ converges for $k > \sigma_D^{(\alpha,\beta)}[f]_g$ and diverges for $k < \sigma_D^{(\alpha,\beta)}[f]_g$.*

Analogously, one can introduced the definition of generalized relative Gol'dberg weak type (α, β) denoted by $\tau_D^{(\alpha,\beta)}[f]_g$ of an entire function $f(z)$ with respect to another entire function $g(z)$ with finite positive generalized relative Gol'dberg lower order (α, β) denoted by $\lambda^{(\alpha,\beta)}[f]_g$ in the following way:

Definition 5.2.4 *Let $f(z)$ and $g(z)$ be any two entire functions of n complex variables. The generalized relative Gol'dberg weak type (α, β) of entire function $f(z)$ with respect to the entire function $g(z)$ having finite positive generalized relative Gol'dberg lower order (α, β) as $\lambda^{(\alpha,\beta)}[f]_g$ $\left(0 < \lambda^{(\alpha,\beta)}[f]_g < \infty\right)$ is defined as :*

$$\tau_D^{(\alpha,\beta)}[f]_g = \sup\left\{ \begin{array}{c} \phi > 0 : M_{f,D}(R) < M_{g,D}\left[\alpha^{-1}\log\left(\phi\left(\exp(\beta(R))\right)^{\lambda^{(\alpha,\beta)}[f]_g}\right)\right] \\ \text{for all } R > R_0(\phi) > 0 \end{array} \right\}$$

$$= \liminf_{R\to\infty} \frac{\exp(\alpha(M_{g,D}^{-1}(M_{f,D}(R))))}{(\exp(\beta(R)))^{\lambda^{(\alpha,\beta)}[f]_g}}.$$

The above definition can also alternatively defined as:

Definition 5.2.5 *Let $f(z)$ and $g(z)$ be any two entire functions of n complex variables having finite positive generalized relative Gol'dberg lower order (α, β) as $\lambda^{(\alpha,\beta)}[f]_g$ $\left(0 < \lambda^{(\alpha,\beta)}[f]_g < \infty\right)$, then the generalized relative Gol'dberg weak type (α, β) denoted by $\tau_D^{(\alpha,\beta)}[f]_g$ of entire function $f(z)$ with respect to the entire function $g(z)$ is defined as:*

$$\text{The integral} \int_{R_0}^{\infty} \frac{\exp^{[2]}(\alpha(M_{g,D}^{-1}(M_{f,D}(R))))}{\left[\exp\left((\exp(\beta(R)))^{\lambda^{(\alpha,\beta)}[f]_g}\right)\right]^{k+1}} dR \, (R_0 > 0)$$

converges for $k > \tau_D^{(\alpha,\beta)}[f]_g$ and diverges for $k < \tau_D^{(\alpha,\beta)}[f]_g$.

Remark 5.2.1 *As Gol'dberg has shown that (see [2]) Gol'dberg type depends on the domain D, so in general all the growth indicators defined in Definition 5.2.2 and Definition 5.2.4 also depend on D.*

Now a question may arise about the equivalence of the definitions of generalized relative Gol'dberg type (α, β) and generalized relative Gol'dberg weak type (α, β) with their integral representations. In the next section we would like to establish such equivalence of Definition 5.2.2 and Definition 5.2.3, and Definition 5.2.4 and Definition 5.2.5 and also investigate some growth properties related to generalized relative Gol'dberg type (α, β) and generalized relative Gol'dberg weak type (α, β) of entire function of n complex variables with respect to another entire function of n complex variables.

Chapter 6

Derivation of some inequalities using generalized relative Gol'dberg type (α, β) and generalized relative Gol'dberg weak type (α, β) of entire functions of several complex variables

Abstract: In this chapter,we establish some important relations relating to generalized relative Gol'dberg type and weak type (α, β) with generalized Gol'dberg type and weak type (α, β) of entire functions of n complex variables, where α, β are continuous non-negative functions defined on $(-\infty, +\infty)$.
Keywords: Generalized Gol'dberg order (α, β), generalized Gol'dberg lower order (α, β), generalized Gol'dberg type (α, β), generalized Gol'dberg weak type (α, β), generalized relative Gol'dberg order (α, β), generalized relative Gol'dberg lower order (α, β), generalized relative Gol'dberg type (α, β), generalized relative Gol'dberg weak type (α, β), increasing function.
Mathematics Subject Classification (2010) : 32A15.

6.1 Introduction.

The relative growth indicators gives a quantitative assessment of how different functions scale each other and until what extent they are self-similar in growth. The concepts of generalized relative Gol'dberg type (α, β) and generalized relative Gol'dberg weak type (α, β) of entire functions of n complex variables are not at all known to the researchers of this area. Therefore the studies of the growths of entire functions of n complex variables in the light of their generalized relative Gol'dberg type (α, β) and generalized relative

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Gol'dberg weak type (α,β) are the prime concern of this chapter. Actually in this chapter we study some relative growth rates of entire functions of n complex variables with respect to another entire function of n complex variables on the basis of their generalized relative Gol'dberg type (α,β) and generalized relative Gol'dberg weak type (α,β). In this present chapter α,β and γ always denote the functions belonging to L^0.

6.2 Lemmas.

From the conclusion of Theorem 4.2.1, we present the following two lemmas which will be needed in the sequel.

Lemma 6.2.1 *Let $f(z)$ and $g(z)$ be two entire functions of n complex variables such that $0<\rho^{(\gamma,\beta)}[f]<\infty$ and $0<\lambda^{(\gamma,\alpha)}[g]=\rho^{(\gamma,\alpha)}[g]<\infty$. Then*

$$\rho^{(\alpha,\beta)}[f]_g=\frac{\rho^{(\gamma,\beta)}[f]}{\rho^{(\gamma,\alpha)}[g]} \quad and \quad \lambda^{(\alpha,\beta)}[f]_g=\frac{\lambda^{(\gamma,\beta)}[f]}{\lambda^{(\gamma,\alpha)}[g]}.$$

Lemma 6.2.2 *Let $f(z)$ and $g(z)$ be two entire functions of n complex variables such that $0<\lambda^{(\gamma,\beta)}[f]=\rho^{(\gamma,\beta)}[f]<\infty$ and $0<\rho^{(\gamma,\alpha)}[g]<\infty$. Then*

$$\rho^{(\alpha,\beta)}[f]_g=\frac{\lambda^{(\gamma,\beta)}[f]}{\lambda^{(\gamma,\alpha)}[g]} \quad and \quad \lambda^{(\alpha,\beta)}[f]_g=\frac{\rho^{(\gamma,\beta)}[f]}{\rho^{(\gamma,\alpha)}[g]}.$$

6.3 Main Results.

In this section we state the main results of the chapter.

Theorem 6.3.1 *Let $f(z)$ and $g(z)$ be two entire functions of n complex variables such that $0<\rho^{(\gamma,\beta)}[f]<\infty$ and $0<\lambda^{(\gamma,\alpha)}[g]=\rho^{(\gamma,\alpha)}[g]<\infty$. Then*

$$\left[\frac{\overline{\sigma}_D^{(\gamma,\beta)}[f]}{\sigma_D^{(\gamma,\alpha)}[g]}\right]^{\frac{1}{\rho^{(\gamma,\alpha)}[g]}} \leq \overline{\sigma}_D^{(\alpha,\beta)}[f]_g \leq \min\left\{\left[\frac{\overline{\sigma}_D^{(\gamma,\beta)}[f]}{\overline{\sigma}_D^{(\gamma,\alpha)}[g]}\right]^{\frac{1}{\rho^{(\gamma,\alpha)}[g]}}, \left[\frac{\sigma_D^{(\gamma,\beta)}[f]}{\sigma_D^{(\gamma,\alpha)}[g]}\right]^{\frac{1}{\rho^{(\gamma,\alpha)}[g]}}\right\}$$
$$\leq \max\left\{\left[\frac{\overline{\sigma}_D^{(\gamma,\beta)}[f]}{\overline{\sigma}_D^{(\gamma,\alpha)}[g]}\right]^{\frac{1}{\rho^{(\gamma,\alpha)}[g]}}, \left[\frac{\sigma_D^{(\gamma,\beta)}[f]}{\sigma_D^{(\gamma,\alpha)}[g]}\right]^{\frac{1}{\rho^{(\gamma,\alpha)}[g]}}\right\} \leq \sigma_D^{(\alpha,\beta)}[f]_g \leq \left[\frac{\sigma_D^{(\gamma,\beta)}[f]}{\overline{\sigma}_D^{(\gamma,\alpha)}[g]}\right]^{\frac{1}{\rho^{(\gamma,\alpha)}[g]}}.$$

Proof. From the definitions of $\sigma_D^{(\gamma,\beta)}[f]$ and $\overline{\sigma}_D^{(\gamma,\beta)}[f]$, we have for all sufficiently large values of R that

$$M_{f,D}(R) \leq \gamma^{-1}\left(\log\left(\left(\sigma_D^{(\gamma,\beta)}[f]+\varepsilon\right)(\exp\beta(R))^{\rho^{(\gamma,\beta)}[f]}\right)\right), \tag{90}$$

$$M_{f,D}(R) \geq \gamma^{-1}\left(\log\left(\left(\overline{\sigma}_D^{(\gamma,\beta)}[f]-\varepsilon\right)(\exp\beta(R))^{\rho^{(\gamma,\beta)}[f]}\right)\right) \tag{91}$$

and also for a sequence of values of R tending to infinity we get that

$$M_{f,D}(R) \geq \gamma^{-1}\left(\log\left(\left(\sigma_D^{(\gamma,\beta)}[f]-\varepsilon\right)(\exp\beta(R))^{\rho^{(\gamma,\beta)}[f]}\right)\right), \tag{92}$$

$$M_{f,D}(R) \leq \gamma^{-1}\left(\log\left(\left(\overline{\sigma}_D^{(\gamma,\beta)}[f]+\varepsilon\right)(\exp\beta(R))^{\rho^{(\gamma,\beta)}[f]}\right)\right). \tag{93}$$

Similarly from the definitions of $\sigma_D^{(\gamma,\alpha)}[g]$ and $\overline{\sigma}_D^{(\gamma,\alpha)}[g]$ it follows for all sufficiently large values of R that

$$M_{g,D}^{-1}(R) \leq \gamma^{-1}\left(\log\left(\left(\sigma_D^{(\gamma,\alpha)}[g]+\varepsilon\right)(\exp(\alpha(R)))^{\rho^{(\gamma,\alpha)}[g]}\right)\right)$$

$$i.e.,\ R \leq M_{g,D}^{-1}\left(\gamma^{-1}\left(\log\left(\left(\sigma_D^{(\gamma,\alpha)}[g]+\varepsilon\right)(\exp(\alpha(R)))^{\rho^{(\gamma,\alpha)}[g]}\right)\right)\right)$$

$$i.e.,\ M_{g,D}^{-1}(R) \geq \alpha^{-1}\left(\log\left(\frac{\exp(\gamma(R))}{\left(\sigma_D^{(\gamma,\alpha)}[g]+\varepsilon\right)}\right)^{\frac{1}{\rho^{(\gamma,\alpha)}[g]}}\right), \tag{94}$$

$$M_{g,D}^{-1}(R) \geq \gamma^{-1}\left(\log\left(\left(\overline{\sigma}_D^{(\gamma,\alpha)}[g]-\varepsilon\right)(\exp\alpha(R)))^{\rho^{(\gamma,\alpha)}[g]}\right)\right)$$

$$i.e.,\ R \geq M_{g,D}^{-1}\left(\gamma^{-1}\left(\log\left(\left(\overline{\sigma}_D^{(\gamma,\alpha)}[g]-\varepsilon\right)(\exp\alpha(R)))^{\rho^{(\gamma,\alpha)}[g]}\right)\right)\right)$$

$$i.e.,\ M_{g,D}^{-1}(R) \leq \alpha^{-1}\left(\log\left(\frac{\exp(\gamma(R))}{\left(\overline{\sigma}_D^{(\gamma,\alpha)}[g]-\varepsilon\right)}\right)^{\frac{1}{\rho^{(\gamma,\alpha)}[g]}}\right) \tag{95}$$

and for a sequence of values of R tending to infinity we obtain that

$$M_{g,D}^{-1}(R) \geq \gamma^{-1}\left(\log\left(\left(\sigma_D^{(\gamma,\alpha)}[g]-\varepsilon\right)(\exp\alpha(R))^{\rho^{(\gamma,\alpha)}[g]}\right)\right)$$

$$i.e.,\ R \geq M_{g,D}^{-1}\left(\gamma^{-1}\left(\log\left(\left(\sigma_D^{(\gamma,\alpha)}[g]-\varepsilon\right)(\exp\alpha(R))^{\rho^{(\gamma,\alpha)}[g]}\right)\right)\right)$$

$$i.e.,\ M_{g,D}^{-1}(R) \leq \alpha^{-1}\left(\log\left(\frac{\exp(\gamma(R))}{\left(\sigma_D^{(\gamma,\alpha)}[g]-\varepsilon\right)}\right)^{\frac{1}{\rho^{(\gamma,\alpha)}[g]}}\right), \tag{96}$$

$$M_{g,D}^{-1}(R) \leq \gamma^{-1}\left(\log\left(\left(\overline{\sigma}_D^{(\gamma,\alpha)}[g]+\varepsilon\right)(\exp\alpha(R))^{\rho^{(\gamma,\alpha)}[g]}\right)\right)$$

$$i.e.,\ R \leq M_{g,D}^{-1}\left(\gamma^{-1}\left(\log\left(\left(\overline{\sigma}_D^{(\gamma,\alpha)}[g]+\varepsilon\right)(\exp\alpha(R))^{\rho^{(\gamma,\alpha)}[g]}\right)\right)\right)$$

$$i.e.,\ M_{g,D}^{-1}(R) \geq \alpha^{-1}\left(\log\left(\left(\frac{\exp(\gamma(R))}{\left(\overline{\sigma}_D^{(\gamma,\alpha)}[g]-\varepsilon\right)}\right)^{\frac{1}{\rho^{(\gamma,\alpha)}[g]}}\right)\right). \tag{97}$$

Chapter 7

Generalized relative Gol'dberg order (α, β) and generalized relative Gol'dberg type (α, β) based growth measure of entire functions of several complex variables

Abstract: In this chapter, we intend to find out generalized relative Gol'dberg order (α, β), generalized relative Gol'dberg type (α, β) and generalized relative Gol'dberg weak type (α, β) of an entire function f of several complex variables with respect to another entire function g of several complex variables when generalized relative Gol'dberg order (γ, β), generalized relative Gol'dberg type (γ, β) and generalized relative Gol'dberg weak type (γ, β) of f and generalized relative Gol'dberg order (γ, α), generalized relative Gol'dberg type (γ, α) and generalized relative Gol'dberg weak type (γ, α) of g with respect another entire function h of several complex variables are given, where α, β, γ are continuous non-negative functions defined on $(-\infty, +\infty)$.
Keywords: Increasing function, generalized relative Gol'dberg order (α, β), generalized relative Gol'dberg type (α, β), generalized relative Gol'dberg weak type (α, β).
Mathematics Subject Classification (2010) : 32A15.

7.1 Introduction.

In continuation of the discussion of previous chapter, question may arise about the values of generalized relative Gol'dberg order (α, β), generalized relative Gol'dberg type (α, β) and generalized relative Gol'dberg weak type (α, β) of an entire function $f(z)$ of n complex variables with respect to another entire function $g(z)$ of n complex variables when generalized relative Gol'dberg order (γ, β), generalized relative Gol'dberg type (γ, β) and generalized relative Gol'dberg weak type (γ, β) of $f(z)$ and generalized relative Gol'dberg order (γ, α), generalized relative Gol'dberg type (γ, α) and generalized relative Gol'dberg

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weak type (γ,α) of $g(z)$ with respect to another entire function $h(z)$ of n complex variables are given. In this chapter we intend to provide this answer. In this present chapter α,β and γ always denote the functions belonging to L^0.

7.2 Main Results.

In this section we present the main results of the chapter.

Theorem 7.2.1 *Let $f(z)$, $g(z)$ and $h(z)$ be three entire functions of n complex variables such that $0<\lambda^{(\gamma,\beta)}[f]_h\leq\rho^{(\gamma,\beta)}[f]_h<\infty$ and $0<\lambda^{(\gamma,\alpha)}[g]_h\leq\rho^{(\gamma,\alpha)}[g]_h<\infty$. Then*

$$\frac{\lambda^{(\gamma,\beta)}[f]_h}{\rho^{(\gamma,\alpha)}[g]_h}\leq\lambda^{(\alpha,\beta)}[f]_g\leq\min\left\{\frac{\lambda^{(\gamma,\beta)}[f]_h}{\lambda^{(\gamma,\alpha)}[g]_h},\frac{\rho^{(\gamma,\beta)}[f]_h}{\rho^{(\gamma,\alpha)}[g]_h}\right\}$$
$$\leq\max\left\{\frac{\lambda^{(\gamma,\beta)}[f]_h}{\lambda^{(\gamma,\alpha)}[g]_h},\frac{\rho^{(\gamma,\beta)}[f]_h}{\rho^{(\gamma,\alpha)}[g]_h}\right\}\leq\rho^{(\alpha,\beta)}[f]_g\leq\frac{\rho^{(\gamma,\beta)}[f]_h}{\lambda^{(\gamma,\alpha)}[g]_h}.$$

Proof. From the definitions of $\rho^{(\gamma,\beta)}[f]_h$ and $\lambda^{(\gamma,\beta)}[f]_h$, we have for all sufficiently large values of R that

$$\begin{aligned}&M_{h,D}^{-1}(M_{f,D}(R))\leq\gamma^{-1}\left(\rho^{(\gamma,\beta)}[f]_h+\varepsilon\right)\beta(R))\\&\text{i.e., }M_{f,D}(R)\leq M_{h,D}\left(\gamma^{-1}\left(\left(\rho^{(\gamma,\beta)}[f]_h+\varepsilon\right)\beta(R)\right)\right),\end{aligned}\tag{142}$$

$$\begin{aligned}&M_{h,D}^{-1}(M_{f,D}(R))\geq\gamma^{-1}\left(\left(\lambda^{(\gamma,\beta)}[f]_h-\varepsilon\right)\beta(R)\right)\\&\text{i.e., }M_{f,D}(R)\geq M_{h,D}\left(\gamma^{-1}\left(\left(\lambda^{(\gamma,\beta)}[f]_h-\varepsilon\right)\beta(R)\right)\right).\end{aligned}\tag{143}$$

Also for a sequence of values of R tending to infinity, we get that

$$\begin{aligned}&M_{h,D}^{-1}(M_{f,D}(R))\geq\gamma^{-1}\left(\left(\rho^{(\gamma,\beta)}[f]_h-\varepsilon\right)\beta(R)\right)\\&\text{i.e., }M_{f,D}(R)\geq M_{h,D}\left(\gamma^{-1}\left(\left(\rho^{(\gamma,\beta)}[f]_h-\varepsilon\right)\beta(R)\right)\right),\end{aligned}\tag{144}$$
$$\begin{aligned}&M_{h,D}^{-1}(M_{f,D}(R))\leq\gamma^{-1}\left(\left(\lambda^{(\gamma,\beta)}[f]_h+\varepsilon\right)\beta(R)\right)\\&\text{i.e., }M_{f,D}(R)\leq M_{h,D}\left(\gamma^{-1}\left(\left(\lambda^{(\gamma,\beta)}[f]_h+\varepsilon\right)\beta(R)\right)\right).\end{aligned}\tag{145}$$

Similarly from the definitions of $\rho^{(\gamma,\alpha)}[g]_h$ and $\lambda^{(\gamma,\alpha)}[g]_h$, it follows for all sufficiently large values of R that

$$\begin{aligned}&M_{h,D}^{-1}(M_{g,D}(R))\leq\gamma^{-1}\left(\left(\rho^{(\gamma,\alpha)}[g]_h+\varepsilon\right)\alpha(R)\right)\\&\text{i.e., }M_{g,D}(R)\leq M_{h,D}\left(\gamma^{-1}\left(\left(\rho^{(\gamma,\alpha)}[g]_h+\varepsilon\right)\alpha(R)\right)\right)\\&\text{i.e., }M_{h,D}(R)\geq M_{g,D}\left(\alpha^{-1}\left(\frac{\gamma(R)}{(\rho^{(\gamma,\alpha)}[g]_h+\varepsilon)}\right)\right),\end{aligned}\tag{146}$$

$$
\begin{aligned}
M_{h,D}^{-1}(M_{g,D}(R)) &\geq \gamma^{-1}\left(\left(\lambda^{(\gamma,\alpha)}[g]_h - \varepsilon\right)\alpha(R)\right) \\
i.e.,\ M_{g,D}(R) &\geq M_{h,D}\left(\gamma^{-1}\left(\left(\lambda^{(\gamma,\alpha)}[g]_h - \varepsilon\right)\alpha(R)\right)\right) \\
i.e.,\ M_{h,D}(R) &\leq M_{g,D}\left(\alpha^{-1}\left(\frac{\gamma(R)}{(\lambda^{(\gamma,\alpha)}[g]_h - \varepsilon)}\right)\right) \qquad (147)
\end{aligned}
$$

and for a sequence of values of R tending to infinity, we obtain that

$$
\begin{aligned}
M_{h,D}^{-1}(M_{g,D}(R)) &\geq \gamma^{-1}\left(\left(\rho^{(\gamma,\alpha)}[g]_h - \varepsilon\right)\alpha(R)\right) \\
i.e.,\ M_{g,D}(R) &\geq M_{h,D}\left(\gamma^{-1}\left(\left(\rho^{(\gamma,\alpha)}[g]_h - \varepsilon\right)\alpha(R)\right)\right) \\
i.e.,\ M_{h,D}(R) &\leq M_{g,D}\left(\alpha^{-1}\left(\frac{\gamma(R)}{(\rho^{(\gamma,\alpha)}[g]_h - \varepsilon)}\right)\right), \qquad (148)
\end{aligned}
$$

$$
\begin{aligned}
M_{h,D}^{-1}(M_{g,D}(R)) &\leq \gamma^{-1}\left(\left(\lambda^{(\gamma,\alpha)}[g]_h + \varepsilon\right)\alpha(R)\right) \\
i.e.,\ M_{g,D}(R) &\leq M_{h,D}\left(\gamma^{-1}\left(\left(\lambda^{(\gamma,\alpha)}[g]_h + \varepsilon\right)\alpha(R)\right)\right) \\
i.e.,\ M_{h,D}(R) &\geq M_{g,D}\left(\alpha^{-1}\left(\frac{\gamma(R)}{(\lambda^{(\gamma,\alpha)}[g]_h + \varepsilon)}\right)\right). \qquad (149)
\end{aligned}
$$

Now from (144) and in view of (146), we get for a sequence of values of R tending to infinity that

$$
\alpha(M_{g,D}^{-1}(M_{f,D}(R))) \geq \alpha\left(M_{g,D}^{-1}\left(M_{h,D}\left(\gamma^{-1}\left(\left(\rho^{(\gamma,\beta)}[f]_h - \varepsilon\right)\beta(R)\right)\right)\right)\right)
$$

$$
\begin{aligned}
& i.e.,\ \alpha(M_{g,D}^{-1}(M_{f,D}(R))) \\
& \geq \alpha\left(M_{g,D}^{-1}\left(M_{g,D}\left(\alpha^{-1}\left(\frac{\gamma\left(\gamma^{-1}\left(\left(\rho^{(\gamma,\beta)}[f]_h - \varepsilon\right)\beta(R)\right)\right)}{(\rho^{(\gamma,\alpha)}[g]_h + \varepsilon)}\right)\right)\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
i.e.,\ \alpha(M_{g,D}^{-1}(M_{f,D}(R))) &\geq \frac{\left(\rho^{(\gamma,\beta)}[f]_h - \varepsilon\right)}{(\rho^{(\gamma,\alpha)}[g]_h + \varepsilon)}\beta(R) \\
i.e.,\ \frac{\alpha(M_{g,D}^{-1}(M_{f,D}(R)))}{\beta(R)} &\geq \frac{\left(\rho^{(\gamma,\beta)}[f]_h - \varepsilon\right)}{(\rho^{(\gamma,\alpha)}[g]_h + \varepsilon)}.
\end{aligned}
$$

As $\varepsilon\,(>0)$ is arbitrary, it follows that

$$
\begin{aligned}
\limsup_{R\to\infty}\frac{\alpha(M_{g,D}^{-1}(M_{f,D}(R)))}{\beta(R)} &\geq \frac{\rho^{(\gamma,\beta)}[f]_h}{\rho^{(\gamma,\alpha)}[g]_h} \\
i.e.,\ \rho^{(\alpha,\beta)}[f]_g &\geq \frac{\rho^{(\gamma,\beta)}[f]_h}{\rho^{(\gamma,\alpha)}[g]_h}. \qquad (150)
\end{aligned}
$$

Analogously from (143) and in view of (149), it follows for a sequence of values of R tending to infinity that

$$
\alpha(M_{g,D}^{-1}(M_{f,D}(R))) \geq \alpha\left(M_{g,D}^{-1}\left(M_{h,D}\left(\gamma^{-1}\left(\left(\lambda^{(\gamma,\beta)}[f]_h - \varepsilon\right)\beta(R)\right)\right)\right)\right)
$$

Chapter 8

Sum and product theorems depending on the generalized relative Gol'dberg order (α, β) and generalized relative Gol'dberg type (α, β)

Abstract: In this chapter, we proved some results about sum and product theorems depending on the generalized relative Gol'dberg order (α, β) ,generalized relative Gol'dberg lower order (α, β), generalized relative Gol'dberg type (α, β) and generalized relative Gol'dberg weak type (α, β) of entire function of n complex variables with respect to another entire function of n complex variables, where α, β are continuous non-negative functions defined on $(-\infty, +\infty)$.
Keywords: Generalized relative Gol'dberg order (α, β) ,generalized relative Gol'dberg lower order (α, β), generalized relative Gol'dberg type (α, β), generalized relative Gol'dberg weak type (α, β) , increasing function, Property (G), Property (X).
Mathematics Subject Classification (2010) : 32A15.

8.1 Introduction.

First of all, we just recall the following well known inequalities for all sufficiently large R relating to any two entire functions $f_1(z)$ and $f_2(z)$ of n complex variables:

$$M_{f_1 \pm f_2, D}(R) \leq M_{f_1, D}(R) + M_{f_2, D}(R), \tag{170}$$

$$M_{f_1 \pm f_2, D}(R) \geq M_{f_1, D}(R) - M_{f_2, D}(R) \tag{171}$$

and

$$M_{f_1 \cdot f_2, D}(R) \leq M_{f_1, D}(R) \cdot M_{f_2, D}(R) \ . \tag{172}$$

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Now let L be a class of continuous non-negative on $(-\infty,+\infty)$ function α such that $\alpha(x)=\alpha(x_0)\geq 0$ for $x\leq x_0$ with $\alpha(x)\uparrow+\infty$ as $x\to+\infty$. For any $\alpha\in L$, we say that $\alpha\in L^0$, if $\alpha(cx)=(1+o(1))\,\alpha(x)$ as $x_0\leq x\to+\infty$ for each $c\in(0,+\infty)$. Clearly, $L^0\subset L$.

Further detailed investigations on the properties of (p,q)-φ relative Gol'dberg order and the (p,q)-φ relative Gol'dberg lower order have been made in [1]. In this connection we just state the following theorems which are introduced by Datta et al. [1] .

Theorem 8.1.1 *Let us consider $f_1(z)$, $f_2(z)$ and $g_1(z)$ are any three entire functions of n complex variables. Also let at least $f_1(z)$ or $f_2(z)$ is of regular (p,q)-φ relative Gol'dberg growth with respect to $g_1(z)$. Then*

$$\lambda_{g_1}^{(p,q)}(f_1\pm f_2,\varphi)\leq\max\{\lambda_{g_1}^{(p,q)}(f_1,\varphi),\lambda_{g_1}^{(p,q)}(f_2,\varphi)\}.$$

The equality holds when any one of $\lambda_{g_1}^{(p,q)}(f_i,\varphi)>\lambda_{g_1}^{(p,q)}(f_j,\varphi)$ hold with at least $f_j(z)$ is of regular (p,q)-φ relative Gol'dberg growth with respect to $g_1(z)$ where $i,j=1,2$ and $i\neq j$.

Theorem 8.1.2 *Let us consider $f_1(z)$, $f_2(z)$ and $g_1(z)$ are any three entire functions of n complex variables such that $\rho_{g_1}^{(p,q)}(f_1,\varphi)$ and $\rho_{g_1}^{(p,q)}(f_2,\varphi)$ exists. Then*

$$\rho_{g_1}^{(p,q)}(f_1\pm f_2,\varphi)\leq\max\{\rho_{g_1}^{(p,q)}(f_1,\varphi),\rho_{g_1}^{(p,q)}(f_2,\varphi)\}.$$

The equality holds when $\rho_{g_1}^{(p,q)}(f_1,\varphi)\neq\rho_{g_1}^{(p,q)}(f_2,\varphi)$.

Theorem 8.1.3 *Let $f_1(z)$, $g_1(z)$ and $g_2(z)$ be any three entire functions of n complex variables such that $\lambda_{g_1}^{(p,q)}(f_1,\varphi)$ and $\lambda_{g_2}^{(p,q)}(f_1,\varphi)$ exists. Then*

$$\lambda_{g_1\pm g_2}^{(p,q)}(f_1,\varphi)\geq\min\{\lambda_{g_1}^{(p,q)}(f_1,\varphi),\lambda_{g_2}^{(p,q)}(f_1,\varphi)\}.$$

The equality holds when $\lambda_{g_1}^{(p,q)}(f_1,\varphi)\neq\lambda_{g_2}^{(p,q)}(f_1,\varphi)$.

Theorem 8.1.4 *Let $f_1(z)$, $g_1(z)$ and $g_2(z)$ be any three entire functions of n complex variables. Also let $f_1(z)$ is of regular (p,q)-φ relative Gol'dberg growth with respect to at least any one of $g_1(z)$ or $g_2(z)$. Then*

$$\rho_{g_1\pm g_2}^{(p,q)}(f_1,\varphi)\geq\min\{\rho_{g_1}^{(p,q)}(f_1,\varphi),\rho_{g_2}^{(p,q)}(f_1,\varphi)\}.$$

The equality holds when any one of $\rho_{g_i}^{(p,q)}(f_1,\varphi)<\rho_{g_j}^{(p,q)}(f_1,\varphi)$ hold with at least $f_1(z)$ is of regular (p,q)-φ relative Gol'dberg growth with respect to $g_j(z)$ where $i,j=1,2$ and $i\neq j$.

Theorem 8.1.5 *Let $f_1(z)$, $g_1(z)$ and $g_2(z)$ be any three entire functions of n complex variables. Then*

$$\rho_{g_1\pm g_2}^{(p,q)}(f_1\pm f_2,\varphi)\leq\max[\min\{\rho_{g_1}^{(p,q)}(f_1,\varphi),\rho_{g_2}^{(p,q)}(f_1,\varphi)\},\min\{\rho_{g_1}^{(p,q)}(f_2,\varphi),\rho_{g_2}^{(p,q)}(f_2,\varphi)\}]$$

when the following two conditions holds:
(i) $\rho_{g_i}^{(p,q)}(f_1,\varphi) < \rho_{g_j}^{(p,q)}(f_1,\varphi)$ *with at least* $f_1(z)$ *is of regular* (p,q)-φ *relative Gol'dberg growth with respect to* $g_j(z)$ *for* $i = 1, 2,$ $j = 1,2$ *and* $i \neq j$*; and*
(ii) $\rho_{g_i}^{(p,q)}(f_2,\varphi) < \rho_{g_j}^{(p,q)}(f_2,\varphi)$ *with at least* $f_2(z)$ *is of regular* (p,q)-φ *relative Gol'dberg growth with respect to* $g_j(z)$ *for* $i = 1, 2,$ $j = 1,2$ *and* $i \neq j$.
The equality holds when any one of $\rho_{g_1}^{(p,q)}(f_i,\varphi) < \rho_{g_1}^{(p,q)}(f_j,\varphi)$ *and any one of* $\rho_{g_2}^{(p,q)}(f_i,\varphi) < \rho_{g_2}^{(p,q)}(f_j,\varphi)$ *hold simultaneously for* $i = 1,2;$ $j = 1,2$ *and* $i \neq j$.

Theorem 8.1.6 *Let* $f_1(z)$, $g_1(z)$ *and* $g_2(z)$ *be any three entire functions of* n *complex variables. Then*

$$\lambda_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2,\varphi) \geq \min[\max\{\lambda_{g_1}^{(p,q)}(f_1,\varphi), \lambda_{g_1}^{(p,q)}(f_2,\varphi)\}, \max\{\lambda_{g_2}^{(p,q)}(f_1,\varphi), \lambda_{g_2}^{(p,q)}(f_2,\varphi)\}]$$

when the following two conditions holds:
(i) $\lambda_{g_1}^{(p,q)}(f_i,\varphi) > \lambda_{g_1}^{(p,q)}(f_j,\varphi)$ *with at least* $f_j(z)$ *is of regular* (p,q)-φ *relative Gol'dberg growth with respect to* $g_1(z)$ *for* $i = 1, 2,$ $j = 1,2$ *and* $i \neq j$*; and*
(ii) $\lambda_{g_2}^{(p,q)}(f_i,\varphi) > \lambda_{g_2}^{(p,q)}(f_j,\varphi)$ *with at least* $f_j(z)$ *is of regular* (p,q)-φ *relative Gol'dberg growth with respect to* $g_2(z)$ *for* $i = 1, 2,$ $j = 1,2$ *and* $i \neq j$.
The equality holds when any one of $\lambda_{g_i}^{(p,q)}(f_1,\varphi) < \lambda_{g_j}^{(p,q)}(f_1,\varphi)$ *and any one of* $\lambda_{g_i}^{(p,q)}(f_2,\varphi) < \lambda_{g_j}^{(p,q)}(f_2,\varphi)$ *hold simultaneously for* $i = 1,2;$ $j = 1,2$ *and* $i \neq j$.

Theorem 8.1.7 *Let us consider* $f_1(z)$, $f_2(z)$ *and* $g_1(z)$ *are any three entire functions of* n *complex variables. Also let at least* $f_1(z)$ *or* $f_2(z)$ *is of regular* (p,q)-φ *relative Gol'dberg growth with respect to* $g_1(z)$ *and* $g_1(z)$ *satisfy the Property (G). Then*

$$\lambda_{g_1}^{(p,q)}(f_1 \cdot f_2,\varphi) \leq \max\left\{\lambda_{g_1}^{(p,q)}(f_1,\varphi), \lambda_{g_1}^{(p,q)}(f_2,\varphi)\right\}.$$

The equality holds when $f_1(z)$ *and* $f_2(z)$ *satisfy Property (X).*

Theorem 8.1.8 *Let us consider* $f_1(z)$, $f_2(z)$ *and* $g_1(z)$ *are any three entire functions of* n *complex variables such that* $\rho_{g_1}^{(p,q)}(f_1,\varphi)$ *and* $\rho_{g_1}^{(p,q)}(f_2,\varphi)$ *exists and* $g_1(z)$ *satisfy the Property (G). Then*

$$\rho_{g_1}^{(p,q)}(f_1 \cdot f_2,\varphi) \leq \max\{\rho_{g_1}^{(p,q)}(f_1,\varphi), \rho_{g_1}^{(p,q)}(f_2,\varphi)\}.$$

The equality holds when f_1 *and* f_2 *satisfy Property (X).*

Theorem 8.1.9 *Let* $f_1(z)$, $g_1(z)$ *and* $g_2(z)$ *be any three entire functions of* n *complex variables such that* $\lambda_{g_1}^{(p,q)}(f_1,\varphi)$ *and* $\lambda_{g_2}^{(p,q)}(f_1,\varphi)$ *exists and* $g_1 \cdot g_2(z)$ *satisfy the Property (G). Then*

$$\lambda_{g_1 \cdot g_2}^{(p,q)}(f_1,\varphi) \geq \min\{\lambda_{g_1}^{(p,q)}(f_1,\varphi), \lambda_{g_2}^{(p,q)}(f_1,\varphi)\}.$$

The equality holds when $g_1(z)$ *and* $g_2(z)$ *satisfy Property (X).*

Conclusion

This book is mainly focused on some growth properties of entire functions of several complex variables, which covers the important branch of complex analysis specially the theory of analytic functions of several variables. All the Chapters of this book deals with some growth properties of entire functions of n complex variables, with the generalization of Gol'dberg order, relative Gol'dberg order, Gol'dberg type and relative Gol'dberg type etc. after introducing non-negative continuous functions α and β defined on $(-\infty, +\infty)$. This book opens the new era of future research. Also the concept of generalized Gol'dberg order and generalized Gol'dberg type should have a broad range of applications in complex dynamics, factorization theory of entire functions of several complex variables, the solution of complex differential equations etc.

During previous decades, several authors made closed investigations on the growth properties of entire functions of several complex variables using different growth indicators such as Gol'dberg order, (p, q)-th Gol'dberg order, relative Gol'dberg order etc. In this book we wish to establish some basic growth properties of entire functions of several complex variables on the basis of their generalized Gol'dberg order (α, β), generalized relative Gol'dberg order (α, β), generalized Gol'dberg type (α, β), generalized relative Gol'dberg type (α, β) where α and β continuous non-negative functions defined on $(-\infty, +\infty)$.We have also discussed about the particular cases when it coincide with present definitions. Integral representations of some definitions are given in some Chapters with some comparative studies.

So, this book (Monograph) will enrich some parts of Pure Mathematics and will give some scopes of study for the future researchers in this branch of complex analysis.

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