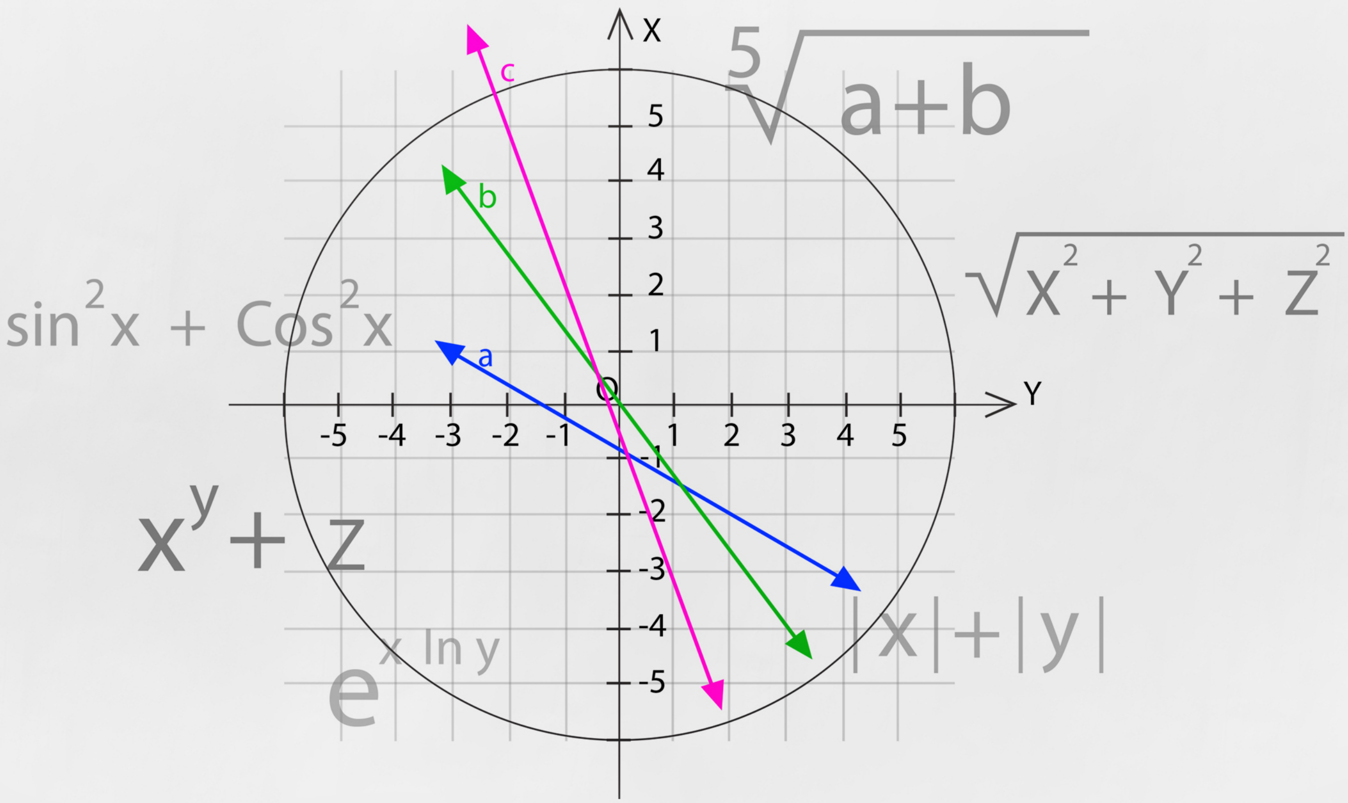
ADVANCES IN SPECIAL FUNCTIONS OF FRACTIONAL CALCULUS:

SPECIAL FUNCTIONS IN FRACTIONAL CALCULUS AND THEIR APPLICATIONS IN ENGINEERING



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Advances in Special Functions of Fractional Calculus: Special Functions in Fractional Calculus and Their Applications in Engineering

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CONTENTS

PREFACE ... i

LIST OF CONTRIBUTORS ... iv

CHAPTER 1 MODIFIED ADAPTIVE SYNCHRONIZATION AND ANTI-SYNCHRONIZATION METHOD FOR FRACTIONAL ORDER CHAOTIC SYSTEMS WITH UNCERTAIN PARAMETERS ... 1

S. K. Agrawa, *Lalit Batra*, *V. Mishra* and *D. Datta*

1. INTRODUCTION ... 1

2. PRELIMINARIES, THEOREM, PROBLEM DESCRIPTION AND CONTROL DESIGN . 5

2.1. Fractional Calculus ... 5

2.2. Problem Description ... 6

2.3. Modified Adaptive Synchronization Controller Des-Ign For Synchronization ... 7

2.4. Modified Adaptive Synchronization Controller Design for Anti-Synchronization ... 7

3. SYSTEM'S DESCRIPTION ... 9

3.1. Fractional Order 3D Autonomous Chaotic System ... 9

3.2. Fractional Order Novel 3d Autonomous Chaotic System ... 10

4. CIRCUIT IMPLEMENTATIONS OF THE CHAOTIC SYSTEM ... 11

4.1. Numerical Simulation and Results ... 13

5. ANTI-SYNCHRONIZATION BETWEEN FRACTIONAL-ORDER 3D AUTONOMOUS AND NOVEL 3D AUTONOMOUS CHAOTIC SYSTEMS USING MODIFIED ADAPTIVE CONTROL METHOD ... 23

5.1. Numerical Simulation and Results ... 24

CONCLUSION ... 33

CONSENT FOR PUBLICATION ... 34

CONFLICT OF INTEREST ... 34

ACKNOWLEDGEMENTS ... 34

REFERENCES ... 34

CHAPTER 2 IMPROVED GENERALIZED DIFFERENTIAL TRANSFORM METHOD FOR A CLASS OF LINEAR NONHOMOGENEOUS ORDINARY FRACTIONAL DIFFERENTIAL EQUATIONS ... 39

İ. Onur KIYMAZ, and *Ayşegül ÇETİNKAYA*

1. INTRODUCTION ... 39

2. IMPROVED GENERALIZED DIFFERENTIAL TRANSFORM METHOD ... 41

3. APPLICATION OF THE METHOD ... 43

CONCLUDING REMARKS AND OBSERVATIONS ... 47

CONSENT FOR PUBLICATION ... 49

CONFLICT OF INTEREST ... 49

ACKNOWLEDGEMENTS ... 49

REFERENCES ... 49

CHAPTER 3 INCOMPLETE K_2-FUNCTION ... 52

Dharmendra Kumar Singh and *Vijay Laxmi Verma*

1. INTRODUCTION ... 52

2. SPECIAL CASES ... 56

3. RELATIONS WITH RIEMANN - LIOUVILLE FRACTIONAL CALCULUS OPERATORS ... 58

CONCLUSION ... 60

CONSENT FOR PUBLICATION ... 61
CONFLICT OF INTEREST ... 61
ACKNOWLEDGEMENTS ... 61
REFERENCES ... 61

CHAPTER 4 SOME RESULTS ON INCOMPLETE HYPERGEOMETRIC FUNCTIONS ... 62
Dharmendra Kumar Singh and *Geeta Yadav*
1. INTRODUCTION ... 62
1.1. Incomplete Hypergeometric Function ... 62
1.2 Incomplete Wright Function ... 63
1.3. Hypergeometric Function ... 64
2. THEOREMS ... 64
CONCLUSION ... 71
CONSENT FOR PUBLICATION ... 71
CONFLICT OF INTEREST ... 72
ACKNOWLEDGEMENTS ... 72
REFERENCES ... 72

CHAPTER 5 TRANSCENDENTAL BERNSTEIN SERIES: INTERPOLATION AND APPROXIMATION ... 73
Z. Avazzadeh, H. Hassani, J.A. Tenreiro Machado, P. Agarwal and *E. Naraghirad*
1. INTRODUCTION ... 73
2. BERNSTEIN POLYNOMIALS ... 74
2.1. Properties of BP ... 75
3. TRANSCENDENTAL BERNSTEIN SERIES ... 76
3.1 Properties of TBS ... 77
3.2. Convergence Analysis ... 79
4. NUMERICAL EXPERIMENTS ... 83
CONCLUSION ... 90
CONSENT FOR PUBLICATION ... 91
CONFLICT OF INTEREST ... 91
ACKNOWLEDGEMENTS ... 91
REFERENCES ... 91

CHAPTER 6 SOME SUFFICIENT CONDITIONS FOR UNIFORM CONVEXITY OF NORMALIZED $_1F_2$ FUNCTION ... 94
Deepak Bansal and *Shilpi Jain*
1. INTRODUCTION ... 94
2. SUFFICIENCY CONDITIONS FOR UNIFORMLY CONVEXITY ... 101
3. SUFFICIENCY CONDITIONS FOR $\Re\left\{{}_1F_2(a,b,c;z)\right\}>1/2$ ... 107
CONSENT FOR PUBLICATION ... 109
CONFLICT OF INTEREST ... 109
ACKNOWLEDGEMENTS ... 109
REFERENCES ... 109

CHAPTER 7 FROM ABEL CONTINUITY THEOREM TO PALEY-WIENER THEOREM ... 112
S. Yu, P. Agarwal and *S. Kanemitsu*
1. INTRODUCTION ... 112
2. PALEY-WIENER THEOREM ... 118
CONSENT FOR PUBLICATION ... 119

CONFLICT OF INTEREST 119
ACKNOWLEDGEMENTS 119
REFERENCES 119

CHAPTER 8 A NEW CLASS OF TRUNCATED EXPONENTIAL-GOULD-HOPPER-BASED GENOCCHI POLYNOMIALS 121
Ghazala Yasmin and *Hibah Islahi*
1. INTRODUCTION 121
2. VARIABLE TRUNCATED EXPONENTIAL-GOULD-HOPPER BASED GENOCCHI POLYNOMIALS 124
3. OPERATIONAL AND INTEGRAL REPRESENTATIONS 128
4. GRAPHICAL REPRESENTATION AND ROOTS 131
CONSENT FOR PUBLICATION 134
CONFLICT OF INTEREST 134
ACKNOWLEDGEMENTS 134
REFERENCES 134

CHAPTER 9 COMPUTATIONAL PRECONDITIONED GAUSS-SEIDEL VIA HALF-SWEEP APPROXIMATION TO CAPUTO'S TIME-FRACTIONAL DIFFERENTIAL EQUATIONS 136
Andang Sunarto, Jumat Sulaiman and Jackel Vui Lung Chew
1. INTRODUCTION 137
2. IMPLICIT APPROXIMATION WITH CAPUTO'S TIME-FRACTIONAL 138
3. STABILITY ANALYSIS 141
4. HALF-SWEEP PRECONDITIONED GAUSS-SEIDEL 143
4.1. Algorithm 1: HSPGS Iterative Method 144
5. NUMERICAL EXPERIMENT 145
CONCLUDING REMARKS 154
CONSENT FOR PUBLICATION 154
CONFLICT OF INTEREST 154
ACKNOWLEDGEMENTS 154
REFERENCES 154

CHAPTER 10 KRASNOSELSKII-TYPE THEOREMS FOR MONOTONE OPERATORS IN ORDERED BANACH ALGEBRA WITH APPLICATIONS IN FRACTIONAL DIFFERENTIAL EQUATIONS AND INCLUSION 157
Nayyar Mehmood and *Niaz Ahmad*
1. INTRODUCTION 157
2. KRASNOSELSKII-TYPE RESULTS FOR TWO MONOTONE SELF OPERATORS 162
3. FIXED POINT RESULTS FOR SET VALUED MONOTONE MAPPINGS 166
3.1. A Generalization of Darbo's Fixed Point Theorem for Monotone Operators 169
3.2. Krasnoselskii-type Results For The Sum Of Two Monotone Multivalued Operators 172
4. APPLICATIONS 173
CONCLUSION 182
CONSENT FOR PUBLICATION 182
CONFLICT OF INTEREST 182
ACKNOWLEDGEMENTS 182
REFERENCES 182

CHAPTER 11 GENERAL FRACTIONAL ORDER QUADRATIC FUNCTIONAL INTEGRAL EQUATIONS: EXISTENCE, PROPERTIES OF SOLUTIONS, AND SOME OF THEIR APPLICATIONS 185

Ahmed M.A. El-Sayed and *Hind H.G. Hashem*
1. INTRODUCTION 185
2. PRELIMINARIES 187
3. EXISTENCE OF SOLUTIONS 191
3.1. Carathéodory Theorem for the Quadratic Integral Equation (1) 191
4. SPECIAL CASES AND REMARKS 196
5. PROPERTIES OF SOLUTIONS 198
5.1. Uniqueness of Solutions of QFIE (1) 199
5.2. Maximal and Minimal Solutions 201
5.3. Comparison Theorem 205
5.4. $\boldsymbol{\delta}$−Approximate Solutions 207
6. APPLICATIONS 212
6.1. Hybrid Functional $\boldsymbol{\phi}$−Differential Equation of Fractional Order 212
6.2. Pantograph Functional $\boldsymbol{\phi}$−Differential Equation of Fractional Order 213
CONCLUSION 214
CONSENT FOR PUBLICATION 214
CONFLICT OF INTEREST 214
ACKNOWLEDGEMENTS 215
REFERENCES 215

CHAPTER 12 NON-LINEAR SET-VALUED DELAY FUNCTIONAL INTEGRAL EQUATIONS OF VOLTERRASTIELTJES TYPE: EXISTENCE OF SOLUTIONS, CONTINUOUS DEPENDENCE AND APPLICATIONS 219
A. M. A. El-Sayed, Sh. M Al-Issa and *Y. M. Y. Omar*
1. INTRODUCTION 219
2. PRELIMINARIES 221
3. SINGLE-VALUED PROBLEM 223
3.1. Existence of Solutions I 223
4. UNIQUENESS OF THE SOLUTION 227
4.1. Continuous Dependence of the Solution 288
4.1.1. Continuous Dependence on the Delay Functions $(\boldsymbol{t})$ 288
4.1.2. Continuous Dependence on the Functions $(\boldsymbol{t},\boldsymbol{s})$ 230
4.2. Application 1 232
5. EXISTENCE OF SOLUTIONS II 233
5.1. Application 2 235
6. SET-VALUED PROBLEM 235
6.1. Existence of Solution 236
6.2. Continuous Dependence on the Set of Selection $\boldsymbol{S_{F1}}$ 237
6.3. Set-valued Chandrasekhar non-linear Quadratic Functional Integral Inclusion 239
6.4. Volterra Integral Inclusion of Fractional Order 239
6.4.1. Differential Inclusion 241
CONCLUSION 241
CONSENT FOR PUBLICATION 241
CONFLICT OF INTEREST 241
ACKNOWLEDGEMENTS 242
REFERENCES 242

CHAPTER 13 CERTAIN SAIGO FRACTIONAL DERIVATIVES OF EXTENDED HYPERGEO-METRIC FUNCTIONS 244
S. Jain, R. Goyal, P. Agarwal and *S. Momani*
1. INTRODUCTION 244

1.1. Main Results 249
1.2. Differential Inclusion 241
CONCLUSION 255
CONSENT FOR PUBLICATION 256
CONFLICT OF INTEREST 256
ACKNOWLEDGEMENTS 256
REFERENCES 256

CHAPTER 14 SOME ERDÉLYI-KOBER FRACTIONAL INTEGRALS OF THE EXTENDED HYPERGEOMETRIC FUNCTIONS 259
S. Jain, R. Goyal, P. Agarwal, Clemente Cesarano4 and *Juan L.G. Guirao*
1. INTRODUCTION AND PRELIMINARIES 259
2. MAIN RESULTS 263
3. SOME SPECIAL CASES OF THE ABOVE FRACTIONAL INTEGRAL FORMULAS 265
CONCLUSION 266
CONSENT FOR PUBLICATION 267
CONFLICT OF INTEREST 267
ACKNOWLEDGEMENTS 267
REFERENCES 267

CHAPTER 15 ON SOLUTIONS OF THE KINETIC MODEL BY SUMUDU TRANSFORM 270
Esra Karatas Akgül, Fethi Bin Muhammed Belgacem and *Ali Akgül*
1. INTRODUCTION 270
2. MATHEMATICAL BACKGROUND 271
3. MAIN RESULTS 274
3.1. Lewis Drying Kinetic Model in Fractional Cases 274
3.1.1. Caputo Fractional Derivative for Drying Kinetic 274
3.1.2. Caputo-Fabrizio Fractional Derivative for Drying Kinetic 275
3.2. Atangan-Baleanu Fractional Derivative for Drying Kinetic 276
3.2.1. Constant Proportional Caputo Fractional Derivative for Drying Kinetic 277
4. NUMERICAL RESULTS 279
CONCLUSION 280
CONSENT FOR PUBLICATION 281
CONFLICT OF INTEREST 281
ACKNOWLEDGEMENTS 281
REFERENCES 281

SUBJECT INDEX 284

PREFACE

In recent years special functions have been developed and applied in a variety of fields, such as combinatory, astronomy, applied mathematics, physics, and engineering due mainly to their remarkable properties. The main purpose of this Special Issue is to be a forum of recently-developed theories and formulas of special functions with their possible applications to some other research areas. This Special Issue provides readers with an opportunity to develop an understanding of recent trends of special functions and the skills needed to apply advanced mathematical techniques to solve complex problems in the theory of partial differential equations. Subject matters are normally related to special functions involving mathematical analysis and its numerous applications, as well as to more abstract methods in the theory of partial differential equations. The main objective of this book is to highlight the importance of fundamental results and techniques of the theory of complex analysis for PDEs, and emphasize articles devoted to the mathematical treatment of questions arising in physics, chemistry, biology, and engineering, particularly those that stress analytical aspects and novel problems and their solutions.

In chapter 1, the authors investigated the Adaptive synchronization and Anti synchronization between fractional-order 3D autonomous chaotic systems and novel 3D autonomous chaotic system with quadratic exponential terms using modified adaptive control method with unknown parameters.

In chapter 2, the authors improved the generalized differential transform method by using the generalized Taylor's formula.

In chapter 3, the authors introduced an incomplete K2-Function.Incomplete hypergeometric function, incomplete hypergeometric function, incomplete confluent hypergeometric function, incomplete Mittag-Leffler function can be deduced as special cases of our findings.

In chapter 4, the authors present some new results for the in-complete hypergeometric function.

In chapter 5, the authors adopt the transcendental Bernstein series (TBS), a set of basis functions based on the Bernstein polynomials (BP), for approximating analytical functions.

In chapter 6, the authors find sufficient conditions under which 1F2 (a; b, c; z) belongsto UCV (α, β) and Sp(α, β). Here, 1F2 (a; b, c; z) is a special case of generalized hypergeometric function for p = 1 and q = 2.

In chapter 7, the authors reveal that the missing link among a few crucial results in analysis, Abel continuity theorem, convergence theorem on (generalized) Dirichlet series, Paley-Wiener theorem is the Laplace transform with Stieltjes integration.

In chapter 8, the authors introduce a hybrid family of truncated exponential-Gould-Hopper based Genocchi polynomials by means of generating function and series definition. Some significant properties of these polynomials are established.

In chapter 9, the authors derived a finite difference approximation equation from the discretization of the one-dimensional linear time-fractional diffusion equations with Caputo's time-fractional derivative.

In chapter 10, the authors derived some important theorems like Krasnoselskii-type Theorems for Monotone Operators in Ordered Banach Algebra with Applications in Fractional Differential Equations and Inclusion.

In chapter 11, authors studied general fractional order quadratic functional integral equations: Existence, properties of solutions and some of their applications.

In chapter 12, the authors consider a nonlinear set-valued delay functional integral equations of Volterra-Stieltjes type.

In chapter 13, the authors establish Saigo fractional derivatives of extended hypergeometric functions. Some special cases of these integrals are also derived.

In chapter 14, the authors establish some new formulas and new results related to the Erdelyi-Kober fractional integral operator which was applied to the extended hypergeometric functions.

In 15 chapter, the authors investigated the kinetic model with four different fractional derivatives. They obtained the solutions of the models by Sumudu transform. They demonstrated results by some figures and prove the accuracy of the Sumudu transform by some theoretical results and applications.

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CHAPTER 1

Modified Adaptive Synchronization and Anti-Synchronization Method for Fractional Order Chaotic Systems with Uncertain Parameters

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Abstract: In the present article, we have investigated the Adaptive synchronization and Anti-synchronization between fractional order 3D autonomous chaotic system and novel 3D autonomous chaotic system with quadratic exponential term using Modified adaptive control method with unknown parameters. The modified adaptive control method is very affective and more convenient in comparison to the existing method for the synchronization of the fractional order chaotic systems. The chaotic attractors and synchronization of the systems are found for fractional order time derivatives described in Caputo sense. Numerical simulation results which are carried out using Adams-Boshforth-Moulton method show that the method is reliable and effective for synchronization and anti-synchronization of autonomous chaotic systems.

Keywords: Modified Adaptive control method, Synchronization and Anti-Synchronization; Fractional derivative, 3D autonomous chaotic systems, Unknown parameters.

1. INTRODUCTION

Nowadays, fractional order derivative has become a popular field of research since fractional order system response ultimately converges to the integer order system. For high accuracy, fractional derivatives are used to describe the dynamics of systems. The attribute of fractional order systems for which they have gained

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popularity in the investigation of dynamical systems is that they allow a greater degree of flexibility in the model. An integer order differential operator is a local operator. Whereas the fractional order differential operator is non-local in that it considers that the future state not only depends upon the present state but also upon all of the history of its previous states. For this realistic property, the usage of fractional order systems is becoming popular. Fractional differential equations have garnered a lot of attention and appreciation recently due to their ability to provide an exact description of different nonlinear phenomena. The advantage of fractional order systems is that they allow greater flexibility in the model. Another advantage of fractional order systems is that they possess memory and display much more sophisticated dynamics compared to its integral order counterpart, which is of great significance in secure communication and control processes. The applications of fractional calculus are growing rapidly. During the last few years, the applications can be found in the fields of science and engineering, including Fluid Mechanics [1, 2], Quantum Mechanics [3], Material Science [4], Viscoelasticity [5], Bioengineering [6], Medicine [7], Biological models [8, 9], Cardiac Tissues [10], *etc*. Analysis of fractional order dynamical systems involving Riemann-Liouville as well as Caputo derivatives have been found in the study [11-13]. The field of chaos in nonlinear dynamics has grabbed the attention of researchers, and this contributes to a significant amount of ongoing research these days.

Synchronization of chaos is a naturally occurring phenomenon where one chaotic dynamical system mimics the dynamical behavior of another chaotic system. This phenomenon can be used in a chaotic communication system as a mechanism for information decoding of the dynamical system. The application of nonlinear dynamical systems has nowadays spread to a wide spectrum of disciplines, including science, engineering, biology, sociology, *etc*. In nonlinear systems, a small change in a parameter in system parameters can lead to sudden and dramatic changes in both the qualitative and quantitative behavior. The idea of synchronizing chaotic systems was introduced by Pecora and Carroll [14] in 1990. They showed that it was possible to synchronize several chaotic systems through a simple coupling. Synchronization of chaotic dynamical systems has been extensively studied by many researchers [15-17] due to its important applications in an ecological system [18], physical system [19], chemical system [20], modeling brain activity, system identification, pattern recognition phenomena and secure communications [21, 22] and so on.

In recent years, several different types of synchronization schemes have been proposed. These include a nonlinear time-delay feedback approach [23], adaptive control [24-26], active control [27-29], sliding mode control [30, 31] and so on. The concept of synchronization can be extended to complete synchronization [32, 33], phase synchronization [34], projective synchronization [35, 36] and function projective synchronization [37, 38].

The synchronization of chaotic systems is a difficult problem due to their extremely sensitive dependence on initial conditions. Any initial correlations present between identical and non-identical systems, starting from very close initial conditions, exponentially decrease to zero with time. Thus, for all practical purposes, any initial synchronization between the systems is bound to disappear rapidly.

The important feature of the study of synchronization is where the difference of states of chaotic systems converge to zero for a long time. This phenomenon is known as complete synchronization. Mathematically, the synchronization is achieved when $\lim_{t\to\infty} \|x_1(t) - x_2(t)\| = 0$, where $x_1(t)$ and $x_2(t)$ are the state vectors of the drive and response systems, respectively. The phenomenon of anti-synchronization is also observed in periodic, chaotic systems. This is a phenomenon in which the state variables of synchronized systems with different initial values have the same absolute but opposite signs. The sum of the two signals is expected to converge to zero when anti-synchronization occurs. Mathematically, the anti-synchronization is achieved when $\lim_{t\to\infty} \|x_1(t) + x_2(t)\| = 0$.

Adaptive Control methods are the control scheme used by a controller which must adapt to a controlled system with parameters that vary from time to time. In practical situations, these parameters may be unknown or initially uncertain. Thus the derivation of adaptive controller for the synchronization of chaotic systems in the presence of system parameter uncertainty is an important problem. This technique is used when the system parameters are unknown. In an adaptive method, a control law and a parameter update rule for unknown parameters are designed in such a way that the chaotic drive system controls the chaotic response system. Most of the studies in synchronization/anti-synchronization involve two identical/non-identical systems under the hypotheses that all the parameters of the master and slave systems are known prior, a controller is constructed with the known parameters and systems are free from external perturbations. But in practical situations, the uncertainties like parameter mismatch and external disturbances may destroy the synchronization and even break it. Therefore, it is necessary to design an adaptive controller and parameter update law for control and synchronization of chaotic systems consisting of unknown parameters to get rid of internal and external noises. In the presence of model uncertainties and external disturbances, an

CHAPTER 2

Improved Generalized Differential Transform Method for a Class of Linear Nonhomogeneous Ordinary Fractional Differential Equations

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Abstract: In this paper, by using the generalized Taylor's formula we improved the generalized differential transform method, which is a useful tool for getting the approximate analytic solutions of fractional differential equations. With this improvement, solutions of a class of linear nonhomogeneous ordinary fractional differential equations, which could not be solved with generalized differential transform method before, will be achieved and the solutions obtained will contain more integers and fractional exponents

Keywords: Fractional Differential Equations, Generalized Taylor's Formula, Generalized Differential Transform Method. 2000 MSC: 65L05, 26A33.

1. INTRODUCTION

In 1986, Zhou [1] presented the concept of differential transformation and used it for obtaining the solutions of linear and non-linear initial value problems in electric circuit analysis. The concept is derived from the Taylor series expansion.

In 1999, Chen and Ho [2] proposed a new transformation for solving partial differential equations, which called two-dimensional differential transform.

In 2008, Momani and Odibat [3] developed this method for finding the solution of linear fractional partial differential equations. This method is based on the generalized Taylor's formula which given by Odibat and Shawagfeh [4] in 2007 and called as generalized differential transform method (GDTM). Many other

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Praveen Agarwal & Shilpi Jain (Eds.)

authors used this method for solving fractional partial differential equations (see for example [5 - 9]).

Recently, El-Ajou *et al.* [10] (see also [11]), introduced a general form of fractional power series.

$$\sum_{n=0}^{\infty}\sum_{j=0}^{m-1} a_{jn}(x-x_0)^{j+n\alpha}, \tag{1.1}$$

where a_{jn}'s are constants, $m \in \mathbb{N}$, $x \geq x_0$ and $0 \leq m-1 < \alpha \leq m$. They also obtained a general form of the generalized Taylor's formula.

$$f(x) = \sum_{n=0}^{\infty}\sum_{j=0}^{m-1} \frac{D^j \boldsymbol{D}^{n\alpha} f(x_0)}{\Gamma(j+n\alpha+1)}(x-x_0)^{j+n\alpha}, \tag{1.2}$$

where $D^j = \frac{d^j}{dx^j}$, $\boldsymbol{D}^{n\alpha} = \boldsymbol{D}^{\alpha}\boldsymbol{D}^{\alpha}\cdots\boldsymbol{D}^{\alpha}$ (n-times), and $\boldsymbol{D}^{\alpha}$ is the usual Caputo fractional derivative [12] which given for $m-1 < \alpha < m$.

$$\boldsymbol{D}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)}\int_0^x \frac{d^m}{dt^m} f(t)(x-t)^{m-\alpha-1}\,dt,$$

and for $\alpha = m \in \mathbb{N}$, $\boldsymbol{D}^{\alpha} f(x) = D^m f(x)$.

Our motivation in this work is to improve the GDTM by moving from the above-mentioned Taylor's formula (2). With the proposed method, more comprehensive solutions, which contains both integer and fractional orders of the unknown function, can be obtained for the following linear non homogenous fractional boundary value problem.

$$\begin{gathered} \boldsymbol{D}^{\alpha} u(x) \pm \lambda u(x) = f(x) \\ u^{(j)}(0) = c_j \end{gathered} \tag{1.3}$$

where $x \geq 0$, $\lambda > 0, c_j \in \mathbb{R}$, $j = 0,1,2 \ldots, m-1$ and $f(x)$ is an analytic function in its domain.

2. IMPROVED GENERALIZED DIFFERENTIAL TRANSFORM METHOD

Before the definition of the improved generalized differential transform method (IGDTM), we gave the following theorem and a basic identity for Caputo fractional derivative in case the reader does not have enough information about fractional derivatives.

Theorem 2.1. Suppose that $f(x) = x^{\lambda} g(x)$, where $\lambda > 1$ and $g(x)$ has the generalized power series expansion $g(x) = \sum_{n=0}^{\infty} a_n x^{\alpha n}$ with radius of convergence $R > 0, 0 < \alpha \leq 1$. If

a) $\beta < \lambda + 1$ and α arbitrary or

b) $\beta \geq \lambda + 1, \alpha$ arbitrary and $a_n = 0$ for $n = 0,1, \ldots, m-1$, where $m - 1 < \beta \leq m$,

then we have

$$\boldsymbol{D}^{\gamma} \boldsymbol{D}^{\beta} f(x) = \boldsymbol{D}^{\gamma+\beta} f(x)$$

for all $x \in (0, R)$ [13].

Lemma 2.1. Let $m - 1 < \alpha < m$ and $\lambda > m - 1$ then

$\boldsymbol{D}^{\alpha} x^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} x^{\lambda-\alpha}$.

For more details about fractional calculus, we refer the books [12, 14, 15] to the reader.

Now, suppose that the function $u(x)$ can be represented as

$$u(x) = \sum_{n=0}^{\infty} \sum_{j=0}^{m-1} a_{nj} (x - x_0)^{n\alpha + j} = \sum_{n=0}^{\infty} \sum_{j=0}^{m-1} U_{\alpha}(n, j)(x - x_0)^{n\alpha + j}, \qquad \textbf{(2.1)}$$

where $m - 1 < \alpha \leq m$. Then the one-dimensional improved generalized differential transform (IGDT) of the function $u(x)$ in (4) is given with.

CHAPTER 3

Incomplete K_2-function

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Abstract: This chapter aims to introduce the incomplete K_2-Function. Incomplete hypergeometric function, incomplete confluent hypergeometric function, and incomplete Mittag-Leffler function can be deduced as special cases of our findings. Some fractional integral formulae illustrate various avenues of their applications.

Keywords: Incomplete pochhammer symbol, K_2 function, Incomplete K_2 function, Incomplete hypergeometric function, Incomplete Mittag-Leffler function.

1. INTRODUCTION

In 1993, Miller and Ross [1] introduced a function.

$$E_x[v,a] = \frac{d^{-v}}{dx^{-v}} e^{ax} = x^v e^{ax} \gamma^*(v,ax) = \sum_{n=0}^{\infty} \frac{a^n x^{n+v}}{\Gamma(n+v+1)}, v \in C \qquad \textbf{(1.1)}$$

based on the solution of the functional order initial value problem, where $\gamma^*(v,ax)$ is the incomplete gamma function and divergent for $|x| = 1$ if $1 \leq R(\gamma)$.

An extension of this function was introduced by Sharma and Dhakar [2] in the following form

$$\underset{(v;a)}{\overset{(p;q)}{K_2}}\left(a_1,\ldots,a_p;\quad b_1,\ldots,b_q;x\right) = \underset{(v;a)}{\overset{(p;q)}{K_2}}(x)$$

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$$= \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{a^n x^{n+v}}{\Gamma(n+v+1)} \quad \textbf{(1.2)}$$

$$= 1 + \sum_{n=1}^{\infty} \prod_{i=1}^{n} (a_i)_n / \prod_{j=1}^{n} (b_j)_n \frac{a^n x^{n+v}}{\Gamma(n+v+1)} \quad \textbf{(1.3)}$$

where$v \in C$ and in which no denominator parameter b_j is allowed to be zero or a negative integer. If any parameter a_i in (1.3) is zero or a negative integer, the serves terminate. An application of the elementary ratio test to the power series on the right side of (1.3) shows at once that.

(i) If $p > q + 1$ the series is convergent for all x.

(ii) If $p = q + 1$and $|x| = 1$, the series can converge in some cases. Let $\gamma = \sum_{i=1}^{p} a_i - \sum_{j=1}^{q} b_j$ it can be show that when $p = q + 1$ the series is absolutely convergent for $|x| = 1$if $R(\gamma) < 0$, conditionally convergent for $x = -1$ if $0 \leq R(\gamma) < 1$ and divergent for $|x| = 1$ is $1 \leq R(\gamma)$

Recently, Singh and Porwal [3] introduced the incomplete Mittag-Leffler function with the help of an incomplete Pochhammer Symbol in the following way,

$$E_{(\alpha,\beta)}^{[\delta,k]}(x) \quad = \sum_{n=0}^{\infty} \frac{[\delta;k]_n}{\Gamma(\alpha n+\beta)} \frac{x^n}{n!} \quad \textbf{(1.4)}$$

$$E_{(\alpha,\beta)}^{[\delta,k]}(x) = \sum_{n=0}^{\infty} \frac{(\delta;k)_n}{\Gamma(\alpha n+\beta)} \frac{x^n}{n!}, \quad \textbf{(1.5)}$$

where,

$$\alpha, \beta, \delta \in C; \mathrm{Re}(\alpha) > 0, Re(\beta) > 0, Re(\delta) > 0.$$

For details of the Mittag-Leffler function, see [4 - 8]. Motivated with work [3], now we introduce incomplete K_2-Function in the following form

$$\begin{matrix} (p;q) \\ K_2 \\ (v,a) \end{matrix} \left[(a_1;k) \ldots, a_p; b_1, \ldots, b_q; x\right]$$

$$= \sum_{n=0}^{\infty} \frac{(a_1;k)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{a^n x^{n+v}}{\Gamma(n+v+1)}, v \in C \quad \textbf{(1.6)}$$

and

$$\begin{matrix} (p;q) \\ K_2 \\ (v,a) \end{matrix} \left[(a_1;k) \ldots, a_p; b_1, \ldots, b_q; x\right]$$

$$= \sum_{n=0}^{\infty} \frac{[a_1;k]_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{a^n x^{n+v}}{\Gamma(n+v+1)}, v \in C \quad \textbf{(1.7)}$$

where$v \in C$ and the domain of convergents will be the same as of equations (1.2) and (1.3).

Where $[\lambda; k]_v$ and $(\lambda; k)_v$ represent incomplete Pochhammer Symbol which is introduced by Srivastava *et al.* [9] and defined as follows:

$$(\lambda; k)_v = \frac{\gamma(\lambda+v,k)}{\Gamma(\lambda)}, (\lambda, v \in C; k \geq 0) \quad \textbf{(1.8)}$$

$$[\lambda; k]_v = \frac{\Gamma(\lambda+v,k)}{\Gamma(\lambda)}, (\lambda, v \in C; k \geq 0) \quad \textbf{(1.9)}$$

and these incomplete Pochhammer symbols satisfy the following decomposition relation:

$$(\lambda; k)_v + [\lambda; k]_v = (\lambda)_v; (\lambda, v \in C; k \geq 0) \quad \textbf{(1.10)}$$

where the Pochhammer Symbol $(\lambda)_v (\lambda, v \in C)$ is given, in general, by

$$(\lambda)_v = \frac{\Gamma(\lambda + v)}{\Gamma(\lambda)} = \begin{cases} 1 & (v = 0; \lambda \in C) \\ \lambda(\lambda + 1) \ldots & (\lambda + n - 1)(v \in N; \lambda \in C) \end{cases}$$

If A_p the array of p parameters like $a_1, a_2, \ldots, a_p$. Then the Pochhammer Symbol $(A_p)_n$, and the incomplete Pochhammer Symbols $(A_p; k)_n$ and $[A_p; k]_n$ are defined by:

CHAPTER 4

Some Results on Incomplete Hypergeometric Functions

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Abstract: Hypergeometric functions are extensions and generalizations of the geometric series, and the process of generalization of hypergeometric series started in the 19th century itself. Thus, the subject of hypergeometrics has a rich history and led to renewed interest. Many mathematicians have presented the hypergeometric function in different ways and explained its properties. Recently, Srivastava *et al.* [9] represented hypergeometric functions in different forms with the help of incomplete pochhammer symbols. This paper is an attempt to present some new results for the incomplete hypergeometric function.

Keywords: Generalized incomplete hypergeometric function, incomplete gamma function, incomplete pochhammer symbols, and decomposition formula.

1. INTRODUCTION

1.1. Incomplete Hypergeometric Function

Incomplete hypergeometric function was introduced and studied by H.M. Srivastava and Agarwal [1], p.675, equations (4.1) and (4.2)], and defined as:

$$ {}_p\gamma_q\left[\begin{matrix}(a_1,x),a_2,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\middle|z\right]=\sum_{n=0}^{\infty}\frac{(a_1,x)_n(a_2)_n\cdots(a)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!} \quad \textbf{(1.1)}$$

and

$$ {}_p\Gamma_q\left[\begin{matrix}(a_1,x),a_2\dots,a_p\\ b_1,\dots,b_q\end{matrix}\middle|z\right]=\sum_{n=0}^{\infty}\frac{[a_1,x]_n(a_2)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!}, \quad \textbf{(1.2)}$$

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where $[a_1;x]_v$ and $(a_1;x)_v$ represent incomplete pochhammer symbols which are defined as follows,

$$(\lambda;x)_v = \frac{\gamma(\lambda+v,x)}{\Gamma(\lambda)} \quad (\lambda, v \in C; x \geq 0) \tag{1.3}$$

and

$$[\lambda;x]_v = \frac{\Gamma(\lambda+v,x)}{\Gamma(\lambda)} \quad (\lambda, v \in C; x \geq 0), \tag{1.4}$$

and these incomplete pochhammer symbols $(\lambda;x)_v$ and $[\lambda;x]_v$ satisfy the following decomposition formula

$$(\lambda;x)_v + [\lambda;x]_v = (\lambda)_v \quad (\lambda, v \in C; x \geq 0),$$

Here, the incomplete Gamma functions, $\gamma(s,x)$ and (s,x) , are defined as

$$\gamma(s,x) = \int_0^x t^{s-1} e^{-t} dt \quad (R(s) > 0;\ x \geq 0) \tag{1.5}$$

and

$$\Gamma(s,x) = \int_x^\infty t^{s-1} e^{-t} dt \quad (R(s) > 0;\ x \geq 0 \ \text{when } x = 0), \tag{1.6}$$

and satisfy the following formula

$$\gamma(s,x) + \Gamma(s,x) = \Gamma(s). \quad (R(s) > 0) \tag{1.7}$$

1.2. Incomplete Wright Function

Incomplete Wright function, ${}_p\overline{\Psi}_q$ and ${}_p\underline{\Psi}_q$, was introduced by Singh and Porwal [2] and defined as:

$${}_p\overline{\Psi}_q\left[\begin{matrix}[\alpha_1, A_1, x]\ldots.(\alpha_p, A_p)\\(\beta_1, B_1)\ldots(\beta_q, B_q)\end{matrix}\middle|z\right]$$

$$= \sum_{k=0}^{\infty} \frac{\Gamma(\alpha_1+A_1k,x)\Gamma(\alpha_2+A_2k)\ldots\Gamma(\alpha_p+A_pk)}{\Gamma(\beta_1+B_1k)\Gamma(\beta_2+B_2k)\ldots\Gamma(\beta_q+B_qk)}\frac{z^k}{k!} \tag{1.8}$$

$${}_p\underline{\Psi}_q\left[\begin{matrix}(\alpha_1, A_1, x)\ldots.(\alpha_p, A_p)\\(\beta_1, B_1)\ldots(\beta_q, B_q)\end{matrix}\middle|z\right]$$

$$= \sum_{k=0}^{\infty} \frac{\gamma(\alpha_1+A_1k,x)\Gamma(\alpha_2+A_2k)\ldots\Gamma(\alpha_p+A_pk)}{\Gamma(\beta_1+B_1k)\Gamma(\beta_2+B_2k)\ldots\Gamma(\beta_q+B_qk)}\frac{z^k}{k!}, \tag{1.9}$$

Where, $|\lambda;x)_n| \leq |(\lambda)_n|$ and $|[\lambda;x]_n| \leq |(\lambda)_n|$. Decomposition of (1.8) and (1.9) gives a well-known Wright function ${}_p\Psi_q$ [3 - 7] who presented its asymptotic expansion for a large value of the argument z under the condition.

$$\sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i > -1. \quad \textbf{(1.10a)}$$

If these conditions are satisfied, the series is in (1.8) and (1.9) is convergent for any z ∈ C.

1.3. Hypergeometric Function

The hypergeometric function [8], ${}_2F_1(a,b,c;z)$ is defined as:

$${}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, \quad |z| < 1 \quad \textbf{(1.10b)}$$

Where a, b, c are complex numbers and $c \neq 0, -1, -2, \ldots$ and the generalized hypergeometric function, in a classical sense, has been defined [5] as:

$${}_pF_q\left[\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix} \mid z\right] = {}_pF_q\left[a_1,\ldots,a_p;b_1,\ldots,b_q;z\right]$$

$$= \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k}\frac{z^k}{k!}, \quad \textbf{(1.11)}$$

Where $(a_i)_n = \frac{\Gamma(a_i+n)}{\Gamma(a_i)}$ and no denominator parameter equal to zero or negative integer.

2. THEOREMS

Theorem 2.1.

If $a, b, c \in C$; $Re(a) > 0$, $Re(b) > 0$, $Re(c) > 0$; $\tau \in N$, then

$$c \quad {}_2F_1([a;x],b;c;z) = c \quad {}_2F_1([a;x],b;c+1;z) + z\frac{d}{dz} \quad {}_2F_1([a;x],b;c+1;z) \quad \textbf{(2.1)}$$

Proof. From the left side of equation (2.1)

$$z\frac{d}{dz} \quad {}_2F_1([a;x],b;c+1;z)$$

CHAPTER 5

Transcendental Bernstein Series: Interpolation and Approximation

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Abstract: This paper adopts the transcendental Bernstein series (TBS), a set of basis functions based on the Bernstein polynomials (BP), for approximating analytical functions. The TBS is more accurate than the BP method, particularly in approximating functions including one or more transcendental terms. The numerical results reveal also the applicability and higher computational efficiency of the new approach.

Keywords: Transcendental functions; Bernstein polynomials; Transcendental Bernstein series.

1. INTRODUCTION

Generalized polynomials have been proven to be valuable tools in several areas of mathematics [1–6]. Several applied sciences adopted the Bernstein polynomials (BP) as a powerful practical tool [7–12]. Moreover, BP has an important role in approximation theory. Draganov [13] proved that several forms of the BP with integer coefficients reveal the property of simultaneous approximation, since the BP approximates the functions and its derivatives. Qian *et al.* [14] provided a uniform approximation of polynomials and BP. Javadi *et al.* [15] introduced the

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Praveen Agarwal & Shilpi Jain (Eds.)

shifted orthonormal BP and derived the operational matrices of integration and delays to solve generalized pantograph equations. Chen *et al.* [16] proposed a method for the numerical solution of a class of variable order fractional linear cable equations and obtained two kinds of operational matrices of BP. Acu and Muraru [17] introduced a bivariate generalization of the Bernstein-Schurer Kantorovich operators based on q-integers and discussed a Bohmann-Korovkin-type approximation theorem for these operators. Bataineh [18] used the BP method and its operational matrices to obtain analytical solutions to variational problems. Mirzaee and Hoseini [19] considered a combination of BP and block-pulse functions to approximate the solution of the optimal control problem for systems governed by a class of nonlinear Volterra integral equations. In the area of computer graphics, in the follow-up of Bézier curves and surfaces, the BP can be employed with a high degree of accuracy [20–25]. Sorokina [26] developed Bernstein-Bézier techniques for analyzing polynomial spline fields in n variables. Lewanowicz *et al.* [27] derived a set of recurrence relations satisfied by the Bezier coefficients of dual bivariate BP and proposed an efficient algorithm for evaluating these coefficients. Winkel [28] studied the ā-BP and ā-Béziercurves based on an interpretation of the ā-BPby means of the convolution of parameters. Winkler and Yang [29] described the application of a structure-preserving matrix method to the deconvolution of two BP basis. Aside from computer applications, BP has been adopted in the solution of elliptic and hyperbolic differential equations based on the Galerkin and collocation methods [30–35].

Hereafter, stemming from the BP formalism, the transcendental Bernstein series (TBS) and their properties are discussed. Indeed, as the set of basis functions, the TBS can approximate analytical functions as discussed in the follow-up. In Section 2, we review the definition and some important properties of the BP, which will be used in the next sections. In Section 3, we introduce the TBS and investigate some fundamental properties of the TBS. In addition, we prove two convergence theorems and apply the TBS to approximate analytical functions. In Section 4, we expand four test functions in terms of the TBS, and we investigate the practical efficiency of the method. Section 5 is dedicated to a brief conclusion.

2. BERNSTEIN POLYNOMIALS

To improve the readability of the follow-up, we review the BP definition and some fundamental properties.

Definition 2.1. The BP of degree m are defined by [36, 37].

$$B_{i,m}(t) = \binom{m}{i} t^i (1-t)^{m-i}, \quad 0 \le i \le m, \tag{2.1}$$

Where $\binom{m}{i} = \frac{m!}{i!(m-i)!}$. By using the binomial expansion

$$(1-t)^{m-i} = \sum_{k=0}^{m-i} (-1)^k \binom{m-i}{k} t^k, \tag{2.2}$$

we have the following formula

$$B_{i,m}(t) = \binom{m}{i} t^i (1-t)^{m-i} = \sum_{k=0}^{m-i} (-1)^k \binom{m}{i}\binom{m-i}{k} t^{i+k}. \tag{2.3}$$

In general, a given function u(t) can be approximated by means of the first $m+1$BP as

$$u(t) = \sum_{i=0}^{m} d_i B_{i,m}(t) = D^T \Phi_m(t), \tag{2.4}$$

where $D^T = [d_0 d_1 \ \ldots \ d_m]$. Furthermore, we have

$$\Phi_m(t) = \left[B_{0,m}(t) B_{1,m}(t) \ldots \ B_{m,m}(t)\right]^T = A\, T_m(t), \tag{2.5}$$

$$A = \begin{pmatrix} a_{00} & a_{01} \ldots & a_{0m} \\ a_{10} & a_{11} \ \ldots & a_{1m} \\ \vdots & \vdots \ \vdots & \vdots \\ a_{m0} & a_{m1} \ \ldots & a_{mm} \end{pmatrix}, \ T_m(t) = [1 \ t \ t^2 \ \ldots \ t^m]^T, \tag{2.6}$$

and

$$a_{ij} = \begin{cases} (-1)^{j-i} \binom{m}{i} \binom{m-i}{j-i}, & i \le j, \\ 0, & i > j, \end{cases} \tag{2.7}$$

which represents the matrix form of approximation based on BP.

2.1. Properties of BP

Some relevant properties of BP are listed as follows:

CHAPTER 6

Some Sufficient Conditions for Uniform Convexity of Normalized $1F_2$ Function

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Abstract: The object of this chapter is to find sufficient conditions under which $z_1F_2(a,b,c;z)$ belongs to UCV (α,β) and $S_p(\alpha,\beta)$. Here, ${}_1F_2(a,b,c;z)$ is a special case of generalized hypergeometric function for $p = 1$ and $q = 2$.

Keywords: Analytic function; Univalent, Starlike; Close-to-convex.

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (1.1)$$

analytic in the open unit disk $\mathbb{D} = z: |z| < 1$and $\mathcal{S}$ denote the subclass of $\mathcal{A}$ that are univalent in $\mathbb{D}$ *i.e.*

$$\mathcal{S} = \{f \in \mathcal{A} |\ f\ is\ one - to - one\ in\ \mathbb{D}\}.$$

A set Ω containing origin in the complex plane is called starlike with respect to origin if for any point z in Ω, the line segment joining the origin to z lies interior of Ω. A function f ∈$\mathcal{A}$ that maps unit disk $\mathbb{D}$ onto a starlike domain is called a starlike function and class of such functions is denoted by $\mathcal{S}^*$. Alternatively, a function f

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Praveen Agarwal & Shilpi Jain (Eds.)

$\in\mathcal{A}$ is called starlike (with respect to 0), denoted by f $\in\mathcal{S}^*$ if tw ∈ f(𝔻)for all w ∈ f(𝔻) and t ∈ [0, 1]. For a given 0 ≤ α < 1, a function f $\in\mathcal{A}$ is called a starlike function of order α, denoted by $\mathcal{S}^*$ (α), if

$$\Re\mathrm{e}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \qquad z \in \mathbb{D}.$$

A set Ω is said to be convex if it is starlike with respect to each of its points, that is if the line segment joining any two points of Ω lies entirely in Ω. A function f $\in\mathcal{A}$ is called convex, denoted by f $\in\mathcal{K}$ if f is univalent in 𝔻 and f(𝔻) is a convex domain. For a given 0 ≤ α < 1, a function f $\in\mathcal{A}$ is called convex function of order α, denoted by $\mathcal{K}$ (α), if

$$\Re\mathrm{e}\left(1+\frac{zf''(z)}{f'(z)}\right) > \alpha, \qquad z \in \mathbb{D}.$$

It is well known that $\mathcal{S}^*$ (0) = $\mathcal{S}^*$ and $\mathcal{K}$ (0) = $\mathcal{K}$. We recall [1] that the function zg′(z) is starlike if and only if the function g(z) is convex. In 1916, L. Bieberbach's conjectured that: The coefficient of each function$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, satisfy

$$|a_n| \le n, \qquad \textbf{(1.2)}$$

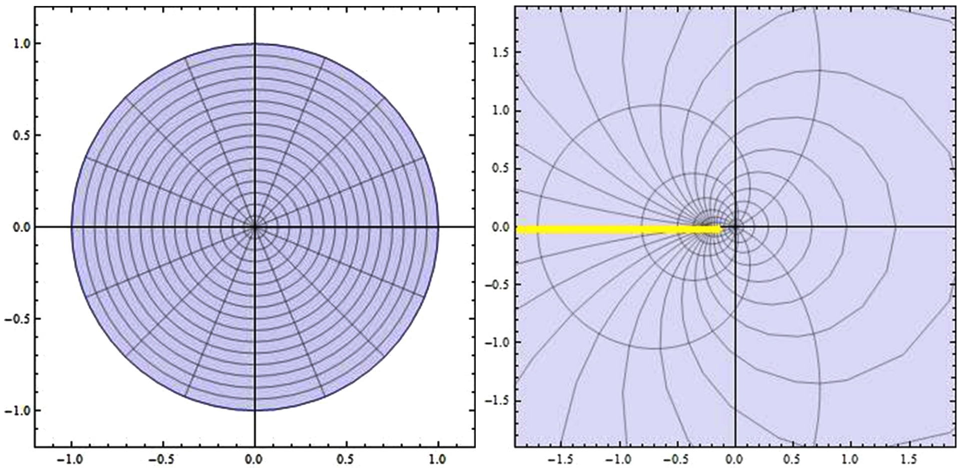
and proved the bound for the case n = 2. In 1985 L. de Branges proved the Biberbach Conjecture with the help of the theory of Special Functions. The equality in (1.2) can be obtained for the Koebe function and its rotations, defined by:

$$K(z) = \frac{z}{(1-z)^2} = z + z^2 + z^3 + \ldots\ldots\ldots\ldots\ldots \qquad \textbf{(1.3)}$$

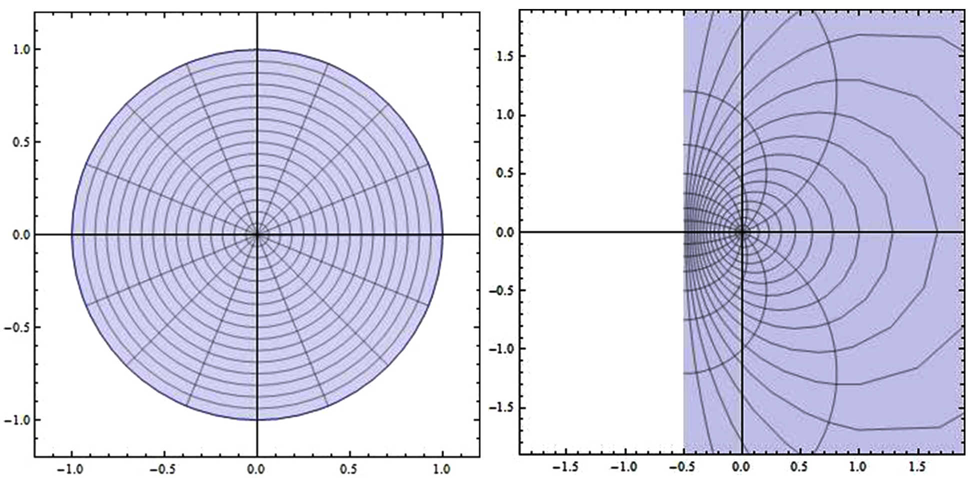
Using mapping properties, one can easily show that the Koebe function maps the unit disk 𝔻 (Fig. (a)) onto ℂ− (−∞, ¼) (Fig. (b)). It is easy to see that K(z) is a starlike but not convex function. The Koebe function plays a role of extremal function for many problems related to the class $\mathcal{S}^*$. Further, the function defined by

$$f_0(z) = \frac{z}{(1-z)} = z + z^2 + z^3 + \ldots\ldots\ldots\ldots\ldots \qquad \textbf{(1.4)}$$

is convex as well as starlike function. This function maps the unit disk 𝔻 (Fig. (c)) onto domain such that $\Re(f_0(z)) > -1/2$ (Fig. (d)). $f_0(z)$ plays a role of extremal function for many problems related to class $\mathcal{K}$.



(a) Unit Disc $\mathbb{D}$ (b) Image of $\mathbb{D}$ under $z/(1-z)^2$



(c) Unit Disc $\mathbb{D}$ (d) Image of $\mathbb{D}$ under $z/(1-z)$

A function f $\in\mathcal{A}$ is said to be convex in the direction of the imaginary axis if f($\mathbb{D}$) intersects every line parallel to the imaginary axis either in an interval or not at all. Given a convex function g $\in\mathcal{K}$ with $g(z) \neq 0$and α < 1, a function f $\in\mathcal{A}$, is called close-to-convex of order α with respect to convex function g, denoted by $C_g(\alpha)$,if

$$\Re\mathfrak{e}\left(\frac{f'(z)}{g'(z)}\right) > \alpha, \qquad z \in \mathbb{D}. \tag{1.5}$$

CHAPTER 7

From Abel Continuity Theorem to Paley-Wiener Theorem

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Abstract: In this note we reveal that the missing link among a few crucial results in analysis, Abel continuity theorem, convergence theorem on (generalized) Dirichlet series, Paley-Wiener theorem is the Laplace transform with Stieltjes integration. By this discovery, the reason why the domains of Stoltz path and of convergence look similar is made clear. Also as a natural intrinsic property of Stieltjes integral, the use of partial summation in existing proofs is elucidated. Secondly, we shall reveal that a basic part of the proof of Paley-Wiener theorem is a version of the Laplace transform.

Keywords: Laplace transform, Stieltjes integral, Abel continuity theorem, Paley-Wiener theorem, conformal mapping, 2010 MSC: 130E99, 44A10, 40A05.

1. INTRODUCTION

Let $\{\lambda_n\}$ be an increasing sequence of real numbers for which we may suppose $\lambda_1 > 0$. For complex coefficients a_n, the series

$$f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \tag{1.1}$$

convergent in some half-plane, is called a generalized Dirichlet series.

1. If $\lambda_n = \log n$ with log denoting the principal value, $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ is (an ordinary) Dirichlet series.

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Praveen Agarwal & Shilpi Jain (Eds.)

2. If $\lambda_n = n$ and $e^{-s} = w$, $f(w) = f(-\log w) = \sum_{n=1}^{\infty} a_n w^n$ is the power series.

In all literature [1], [2], [3], *etc.* the convergence theorem for generalized Dirichlet series, Theorem 1 and the Abel continuity theorem, Corollary 1 are regarded as independent and proofs are given separately. Cf. also [4] (cf. [5]). In [6] it is shown that Theorem 1 entails Corollary 1 *via* a counterpart, Corollary 2 together with conformality of the analytic mapping $e^{-s} = w$, thus revealing the reason why the convergence domains are angular domains of a similar shape. The proof uses a general form of the partial summation [6, Lemma 2] for a generalized sequence $\{\lambda_n\}$, thus unifying all existing proofs.

In this note we employ a general treatment by (Lebesgue-) Stieltjes integrals to attain two objects at a stretch. I.e. we follow [7] to introduce Corollary 3 whose discrete version leads to Theorem 1. In the proof, integration by parts is used which is a more general version of the partial summation. Then on one hand we cover Abel continuity theorem by the convergence theorem, Corollary 3, for Laplace transforms and conformality, revealing the reason why convergence domains being similar.

On the other hand, we shall show that the basic part of the Paley-Wiener theorem (cf. *e.g.* [8]) is laid by the Laplace transform method. Then we appeal to two fundamental results, the Plancherel formula and the Fourier inversion formula to conclude the theorem. By finding this hidden link of Laplace transform, we are able to treat these two remote-looking objects of Paley-Wiener theorem and Abel continuity theorem in a unified way, up to some auxiliary fundamental results. The Paley-Wiener theorem has recently been highlighted in view of its essential application to signal restoration. In both well-known approaches by sampling [7], [10], [11] and by Bernstein polynomials [12] the Paley-Wiener theorem plays a fundamental role. *This is a typical example of ideas indoctrinated previously with the established methods and attitudes of the discipline, can sometimes point to unorthodox, though remarkably simple, solutions to those problems.*

Theorem 1. *If the series* (1.1) *is convergent for* $s = s_0 = \sigma_0 + it_0$, *then f (s) is uniformly convergent in the right half-plane* $\sigma > \sigma_0$ *in the wide sense and represents an analytic function there. More precisely, let D be an angular domain*

$$\sigma - \sigma_0 \geq 0, \quad \arg(s - s_0) \leq \delta \qquad \textbf{(1.2)}$$

with $0<\delta<\frac{\pi}{2}$ $0<\delta<\frac{\pi}{2}$. *Then* $f(s)$ *is uniformly convergent on D in the wide sense.*

Corollary 1. (Abel continuity theorem)

Suppose a power series $f(z)=\sum_{n=1}^{\infty}a_n z^n$ *converges at the point* z_0 *on its circle of convergence. Draw two chords (inside the circle) that start from* z_0 *and form an angle* δ *with the tangent at* z_0 *of the circle* $\left(0<\delta<\frac{\pi}{2}\right)$. *Let* Δ *be the (closure of) intersection of this angular subdomain and the disc of convergence. Then* $f(z)$ *approaches* $f(z_0)$ *as* $z\to z_0$ *in the angular domain inside* Δ. *This is often said as z approaches to* z_0 ***along Stoltz path***.

Corollary 2. (Counterpart of Abel continuity theorem)

f(s) approaches to f(s_0*) as* $s\to s_0$ *in the angular domain* (1.2)

Lemma 1.

(i) *The Stieltjes integral* $\int_a^b f\,\mathrm{d}g$ *exists if f is continuous and g is of bounded variation and linear in f and g. The role can be changed in view of Item* (ii)*. It holds that*

$$\int_a^b \mathrm{d}g(x)=g(b)-g(a). \qquad \textbf{(1.3)}$$

(ii) *The formula for integration by parts holds true:*

$$\int_a^b f(x)\mathrm{d}g(x)=[f(x)g(x)]_a^b-\int_a^b g(x)\mathrm{d}f(x), \qquad \textbf{(1.4)}$$

provided that f is continuous and g is of bounded variation or g is continuous and f is of bounded variation.

CHAPTER 8

A New Class of Truncated Exponential-Gould-Hopper-based Genocchi Polynomials

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Abstract: The present paper introduces a hybrid family of truncated exponential-Gould-Hopper-based Genocchi polynomials by means of generating function and series definition. Some significant properties of these polynomials are established. In addition, graphs of truncated exponential-Gould-Hopper-based Genocchi polynomials are drawn using Matlab. Thereafter, the distribution of zeros of these polynomials is shown.

Keywords: Truncated exponential-Gould-Hopper polynomials, Genocchi polynomials, Monomiality principle, Operational techniques.

1. INTRODUCTION AND PRELIMINARIES

In its various forms, multivariable and generalized forms of the special functions have been an object of speculation and application in recent years. Most of the special functions and their generalizations are suggested by physical problems. We recall the 3-variable truncated exponential-based Gould-Hopper polynomials (3VTEGHP), denoted by ${}_{e^{(r)}}H_n^{(s)}(x,y,z)$, defined by means of the following generating function [1]:

$$\frac{\exp\left(xt+zt^s\right)}{1-yt^r}=\sum_{n=0}^{\infty}{}_{e^{(r)}}H_n^{(s)}\left(x\ y,z\right)\frac{t^n}{n!} \qquad \textbf{(1.1)}$$

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and posses the following equivalent forms of series representation in terms of 2 variable truncated exponential polynomials (2VTEP) [2], denoted by $e_n^{(r)}(x,y)$; Gould-Hopper polynomials (GHP) [3], denoted by $H_n^{(s)}(x,z)$; and in terms of x,y,z:

$$ {}_{e^{(r)}}H_n^{(s)}(x,y,z) = n! \sum_{k=0}^{[\frac{n}{s}]} \frac{z^k\, e_{n-sk}^{(r)}(x,y)}{k!(n-sk)!}, \tag{1.2}$$

$$ {}_{e^{(r)}}H_n^{(s)}(x,y,z) = n! \sum_{m=0}^{[\frac{n}{r}]} \frac{y^m\, H_{n-rm}^{(s)}(x,z)}{(n-rm)!} \tag{1.3}$$

and

$$ {}_{e^{(r)}}H_n^{(s)}(x,y,z) = n! \sum_{k,m=0}^{sk+rm\leq n} \frac{x^{n-sk-rm} y^m z^k}{k!(n-sk-rm)!}, \tag{1.4}$$

respectively.

It is shown in [1] that the 3VTEGHP ${}_{e^{(r)}}H_n^{(s)}(x,y,z)$ are quasimonomial [4, 5], and their multiplicative and derivative operators are given by:

$$ \hat{M}_{e^{(r)}H^{(s)}} = x + ry\partial_y y\partial_x^{r-1} + sz\partial_x^{s-1} \tag{1.5}$$

and

$$ \hat{P}_{e^{(r)}H^{(s)}} = \partial_x, \tag{1.6}$$

respectively.

Now since ${}_{e^{(r)}}H_0^{(s)}(x,y,z) = 1$, so monomiality principle implies that the 3VTEGHP ${}_{e^{(r)}}H_n^{(s)}(x,y,z)$ can be constructed as:

$$ {}_{e^{(r)}}H_n^{(s)}(x,y,z) = \hat{M}_{e^{(r)}H^{(s)}}{}^n\{1\} = \left(x + ry\partial_y y\partial_x^{r-1} + sz\partial_x^{s-1}\right)^n\{1\}, \tag{1.7}$$

which yields the series definition (1.4).

In view of identity (1.7), the exponential generating function of the GHP $H_n^{(s)}(x,y)$ can be given by:

$$\exp(\widehat{M}_{e^{(r)}H^{(s)}}t)\{1\} = \sum_{n=0}^{\infty} {}_{e^{(r)}}H_n^{(s)}(x,y,z)\frac{t^n}{n!}, \quad \textbf{(1.8)}$$

which gives generating function (1.1).

The operational representation of 3VTEGHP ${}_{e^{(r)}}H_n^{(s)}(x,y,z)$ is:

$${}_{e^{(r)}}H_n^{(s)}(x,y,z) = \exp\left(z\partial_x^s + y\partial_y y\partial_x^r\right)\{x^n\}. \quad \textbf{(1.9)}$$

The operational representation which links the 3VTEGHP ${}_{e^{(r)}}H_n^{(s)}(x,y,z)$ with the 2VTEP $e_n^{(r)}(x,y)$ and GHP $H_n^{(s)}(x,y)$ is:

$${}_{e^{(r)}}H_n^{(s)}(x,y,z) = \exp(z\partial_x^s)\{e_n^{(r)}(x,y)\} \quad \textbf{(1.10)}$$

and

$${}_{e^{(r)}}H_n^{(s)}(x,y,z) = \exp\left(y\partial_y y\partial_x^r\right)\{H_n^{(s)}(x,z)\}, \quad \textbf{(1.11)}$$

respectively.

The integral representation for the 3VTEGHP ${}_{e^{(r)}}H_n^{(s)}(x,y,z)$ in terms of 2-iterated Gould-Hopper polynomials (2IGHP) [6] is:

$${}_{e^{(r)}}H_n^{(s)}(x,y,z) = \int_0^{\infty} e^{-u}\, {}_{H^{(r)}}H_n^{(s)}(x,yu,z)du. \quad \textbf{(1.12)}$$

The research on Genocchi numbers and Genocchi polynomials can be traced back to Angelo Genocchi (1817-1889). During these very recent years, Genocchi numbers and Genocchi polynomials are extensively studied in many different contexts in mathematics and physics, such as, elementary number theory, analytic number theory, theory of modular forms, p-adic analytic number theory, different topology, and quantum physics. The generating function of Genocchi polynomials $G_n(x)$ are given by

$$\left(\frac{2t}{e^t+1}\right)\exp(xt) = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}, \quad |t| < \pi. \quad \textbf{(1.13)}$$

The Genocchi numbers G_n are given as

CHAPTER 9

Computational Preconditioned Gauss-Seidel *via* Half-Sweep Approximation to Caputo's Time-Fractional Differential Equations

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Abstract: In this paper, we derived a finite difference approximation equation from the discretization of the one-dimensional linear time-fractional diffusion equations with Caputo's time-fractional derivative. A linear system is generated by implementing Caputo's finite difference approximation equation on the specified solution domain. Then, the linear system is solved using the proposed half-sweep preconditioned Gauss-Seidel iterative method. The effectiveness of the method is studied, and the efficiency is analyzed compared to the existing preconditioned Gauss-Seidel, also known as the full-sweep preconditioned Gauss-Seidel and the classic Gauss-Seidel iterative method. A few examples of the mathematical problem are delivered to compare the performance of the proposed and existing methods. The finding of this paper showed that the proposed method is more efficient and effective than the full-sweep preconditioned Gauss-Seidel and Gauss-Seidel methods.

Keywords: Caputo's fractional derivative, Implicit scheme, Half-sweep, Preconditioned, Gauss-Seidel, Iterative method.

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Praveen Agarwal & Shilpi Jain (Eds.)

1. INTRODUCTION

Fractional partial differential equations (FPDEs) have been actively studied by many researchers nowadays. Several studies illustrated the importance of FPDEs to model complex phenomena [1-4]. The capability of FPDEs to capture anomalous phenomena, which some of the past PDEs have failed, attracted many researchers to apply in natural science fields other than pure mathematics [5-8]. By considering pure mathematics alone, various methods have been proposed to obtain accurate and efficient numerical solutions for the FPDEs [9-12].

The present paper focuses on developing an efficient numerical method to solve one-dimensional linear time-fractional diffusion equations via Caputo's time-fractional derivative. A numerical method is proposed to improve the effectiveness of the preconditioned Gauss-Seidel (GS) iteration studied by [13]. The contribution of the paper is to present the effectiveness of the half-sweep computation approach [14] in deriving a better version of the preconditioned Gauss-Seidel iterative method, which can be named the half-sweep preconditioned Gauss-Seidel (HSPGS) method. The half-sweep computation approach is a computational complexity reduction approach that has been extended from the finite difference method. This approach is capable of reducing the complexity of solving a large system of equations generated from linear and nonlinear PDEs [15-18]. The reported results from [15-18] motivated this study to investigate the efficacy of half-sweep computation in solving an FPDE.

The theory and concept of iterative methods have contributed to numerical analysis and computation since the early 20th century. Several researchers introduced and explained different families of iterative methods [19-21]. Among the families of efficient iterative methods, the preconditioned iterative methods are widely accepted as one of the efficient methods for solving equations [22-25]. Thus, the paper presents the formulation of a preconditioned iterative method using the half-sweep computation approach and Gauss-Seidel iteration. The effectiveness of the proposed HSPGS method is investigated and compared against the existing methods, such as the full-sweep preconditioned Gauss-Seidel [13] and the classic Gauss-Seidel methods.

2. IMPLICIT APPROXIMATION WITH CAPUTO'S TIME-FRACTIONAL

Let us consider a time-FPDE to be defined as

$$\frac{\partial^{\alpha} U(x,t)}{\partial t^{\alpha}} = a(x)\frac{\partial^2 U(x,t)}{\partial x^2} + b(x)\frac{\partial U(x,t)}{\partial x} + c(x)U(x,t), \qquad (\mathbf{1})$$

where $a(x), b(x)$ and $c(x)$ are known functions or constants, whereas α is a parameter that refers to the fractional-order of time derivative. A discrete approximation equation to the mathematical problem shown by equation (1) can be formulated using finite differences and Caputo's time-fractional derivative. Some basic definitions of the fractional derivative theory are used to approximate the fractional derivative in equation (1), which can be stated as [26]:

Definition 1. The Riemann-Liouville fractional integral operator, $J^{\alpha}$of order-α is defined as

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha} f(t)\, dt, \alpha > 0, x > 0. \qquad (\mathbf{2})$$

Definition 2. Caputo's fractional partial derivative operator, D^{α} of order -α is defined as

$$D^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)}\int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}}\, dt, \alpha > 0, \qquad (\mathbf{3})$$

with $m-1 < \alpha \leq m, m \in N, x > 0$.

This paper aims to obtain the numerical solution of equation (1) using the implicit approximation based on Caputo's definition. The solution domain is subjected to Dirichlet boundary conditions, and we consider a nonlocal fractional derivative operator. The paper considers the following general notation of boundary and initial conditions:

$$U(0,t) = g_0(t), U(\ell,t) = g_1(t), U(x,0) = f(x), \qquad (\mathbf{4})$$

CHAPTER 10

Krasnoselskii-type Theorems for Monotone Operators in Ordered Banach Algebra with Applications in Fractional Differential Equations and Inclusion

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Abstract: This chapter discusses Krasnoselskii-type fixed point results for monotone operators. It is well known that the monotone operators are not continuous on the whole domain, so we will find the solutions of discontinuous operator equations and inclusions. The presented fixed point results may be considered as variants of the Krasnoselskii fixed point theorem in a more general setting. The results of Darbo, Schauder and Bohnentblust-Karlin are also generalized. We prove these results for the case of single-valued and set-valued monotone operators. We use our main result for single-valued operators to obtain the existence of solutions of anti-periodic ABC fractional BVP. The fixed point result for set-valued monotone operators is used to discuss the existence of solutions of a given fractional integral inclusion in ordered Banach spaces.

Keywords: Krasnoselskii's fixed point theorem, Set valued mappings, Convex, Compact, Closed sets, Banach spaces, Fractional differential equations, Atangana and Baleanu derivatives.

1. INTRODUCTION

Fractional calculus is one of the most emerging fields of mathematics. Many physical models have been studied more accurately by virtue of fractional calculus. Some classical fractional derivatives are Riemann Liouville, Hadamard, Caputo and Grunwald-Letnikov; all of these have many applications [1]. Many of them have a singular Kernel, which creates some flaws when applied to some physical

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Praveen Agarwal & Shilpi Jain (Eds.)

problems. To overcome this problem, Caputo and Febrizio [2] introduced a new fractional derivative with a non-singular kernel. It also involves some ambiguities that were later removed by Atangana and Baleanu [3] by introducing a more general form of the fractional derivative with a non-singular kernel, using the Mittage-Leffler function. Many applications can be seen in the literature.

Topological fixed point theory is one of the most emerging fields in nonlinear analysis. It was initiated about a century ago. Utilizing the topological methods in the theory of differential equations, Poincare initiated the idea of fixed point theory [4]. After that, the most celebrated fixed point theorem of Brouwer was presented in 1910 [5]. Schauder generalized this result for the case of infinite dimensional Banach spaces [6]. The following is the theorem of Schauder.

Theorem 1 Let δ be a convex and compact subset of a Banach space H. Suppose ξ is a continuous mapping of δ into H. Then ξ possesses at least one fixed point.

The famous Banach contraction principle was proved in 1922 [7], which gave the theory more strength and a new direction to the stability and existence theory of nonlinear operators. This principle is given as follows.

Theorem 2 Let (δ, d) be a complete metric space and suppose ξ is a self-mapping of δ such that

$$d\left(\xi(\lambda),\xi(\mu)\right)\leq hd(\lambda,\mu),\text{ for all }\lambda,\mu\in\delta,$$

For some, $h\in(0,1)$. Then ξ possesses only one fixed point.

While studying the theory of perturbed differential equations and the article of Schauder, Krasnoselskii [8] came to know that the inversion of a perturbed differential equation might have formed a sum of contractive and compact operators. This thought ended with a famous Krasnoselskii's fixed point theorem for the sum of two operators. The result of Krasnoselskii's fixed point theorem is given below.

Theorem 3 Let δ be a nonempty convex and closed subset of a Banach space *H*. Suppose ξ,ζ are mappings of δ into H such that:

(i) $\lambda,\mu\in\delta,$ implies $\xi(\lambda)+\zeta(\mu)\in\delta,$

(ii) ζ is a contraction mapping,

(iii) ξ is continuous and compact.

Then $\xi+\zeta$ possesses at least one fixed point.

Remark 4 If ξ be a zero operator and the condition of convexity is relaxed, then it is Banach contraction theorem. If ζ is a zero operator with the condition of compactness, then it is the fixed point theorem of Schauder.

Many generalizations and extensions of Krasnoselskii's fixed point theorems are presented in the literature [9-13]. Different directions have been considered to generalize this result, for example, generalizing the space, weakening compactness or continuity, or taking set-valued maps instead of single-valued operators, etc.

A novel way of generalization is to use partial ordering on space with some other conditions. This kind of generalization was considered by Ran and Reuring [14], in 2004. Their main result is stated as follows.

Theorem 5 Let the lower or upper bound of a partially ordered set δ exists. Suppose d is a complete metric on δ , and ξ is a monotone and continuous self-mapping on δ ,

(i) $\exists\ h\in(0,1)$ and for all $\lambda\geq\mu$ such that $d\left(\xi(\lambda),\xi(\mu)\right)\leq hd(\lambda,\mu)$;

(ii) $\exists\ \lambda_0\in\delta$ such that either $\lambda_0\leq Q\lambda_0$ or $\lambda_0\geq Q\lambda_0$. Then ξ possesses only one fixed point.

This theorem was applied to matrix equations to develop solvability results for these equations. A modified variant of the above result by weakening the continuity of the mappings was given by Nieto and Rodriguez-Lopez [15] as follows.

Theorem 6 Let the lower or upper bound of a partially ordered set δ exists. Suppose d is a complete metric on δ and ξ is a monotone self-mapping on δ ,

(i) $\exists\ h\in(0,1)$ and for all $\lambda\geq\mu$ such that $d\left(\xi(\lambda),\xi(\mu)\right)\leq hd(\lambda,\mu)$;

(ii) $\exists\ \lambda_0\in\delta$ such that either $\lambda_0\leq Q\lambda_0$ or $\lambda_0\geq Q\lambda_0$;

CHAPTER 11

General Fractional Order Quadratic Functional Integral Equations: Existence, Properties of Solutions, and Some of their Applications

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Abstract: In this chapter, we are interested in a certain class of integral equations, namely the quadratic integral equation. In this case, the unknown function is treated by some operators, then a pointwise multiplication of such operators is applied. The study of such a kind of problem was begun in the early 60's due to the mathematical modeling of radiative transfer. The main objective was to present a special method or technique and results concerning various existence for a certain quadratic integral equation.

Keywords: Quadratic integral equation, Carathéodory Theorem, Continuous solution, Iterative scheme, Maximal and minimal solutions, Comparison Theorem, $\delta-$Approximate solutions, Hybrid functional $\phi-$differential equation, Pantograph functional $\phi-$ differential equation.

1. INTRODUCTION

Quadratic integral equations have many useful applications and problems in the real world. For example, quadratic integral equations are often applicable in the theory of radiative transfer, the kinetic theory of gases, the theory of neutron transport, the queuing theory, and the traffic theory. Many authors studied the existence of solutions for several classes of nonlinear quadratic integral equations (see, *e.g.* [1-13] and [12-28]).

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Let $J = [0, T]$, $\phi: J \to R$ be increasing and absolutely continuous and $\psi_i: J \to J, i = 1,2$ be continuous. Let $\beta \in (0,1)$ and $t \in J$.

Here, we study the existence of continuous solutions $x \in C(J)$ of the $\phi-$ fractional-orders quadratic functional integral equation

$$x(t) = a(t) + f_1\left(t, x\big(\psi_1(t)\big)\right). \int_0^t \frac{(\phi(t)-\phi(s))^{\beta-1}}{\Gamma(\beta)} f_2\left(s, x\big(\psi_2(s)\big)\right)\phi'(s)ds, t \in J,\ \beta \in (0,1]. \quad \textbf{(1)}$$

We discuss some properties of the solutions, prove the existence of maximal and minimal solutions of the quadratic integral equation (1) and introduce some particular cases and applications.

Brownian motion is an anomalous diffusion process driven by a fractional integral equation in the sense of Erdélyi-Kober. For this reason, it is proposed to call such a family of diffusive processes Erdélyi- Kober fractional diffusion [29].

An Erdélyi-Kober operator is a fractional integration operator introduced by Arthur Erdélyi (1940) and Hermann Kober (1940).

The Erdélyi-Kober fractional integral is given by [23-25]

$$I_m^\alpha f(t) = \int_0^t \frac{(t^m - s^m)^{\alpha-1}}{\Gamma(\alpha)}\, m\, s^{m-1}\, f(s)\, ds,\ m > 0, \alpha > 0,$$

which generalizes the Riemann fractional integral (when $m = 1$) and its generalized fractional derivative of order α, like:

$$D_m^\alpha f(t) = D_m I_m^{1-\alpha},\ m > 0,\ \alpha \in (0,1).$$

For the properties of Erdélyi-Kober operators, see [30] and [31] for examples.

As a particular case of equation (1), we can consider the Erdélyi-Kober functional quadratic integral equation

$$x(t) = a(t) + f_1(t, x(\psi(t))) \int_0^t \frac{(t^m - s^m)^{\beta-1}}{\Gamma(\beta)} f_2(s, x(\psi(s)))m\, s^{m-1}ds, t \in J \quad \textbf{(2)}$$

when $\phi(t) = t^m,\ m > 0.$

As applications, we consider the nonlinear hybrid functional ϕ −differential equation of fractional order

$$D_{\phi}^{\beta}\left(\frac{x(t)-a(t)}{f_1(t,\ x(\psi_1(t)))}\right) = f_2(t, x(\psi_2(t))),\ \beta \in (0,1).$$

The importance of the investigations of hybrid differential equations lies in the fact that they include several dynamic systems as special cases.

We also consider the ϕ −differential equation of pantograph-type delay of fractional order

$$D_{\phi}^{\beta}\left(\frac{x(t)-a(t)}{f_1(t,x(\sigma_1 t))}\right) = f_2(t, x(\sigma_2 t)), t \in J,\ \ \beta \in (0,1).$$

where $\sigma_1, \sigma_2 \in (0,1)$.

2. PRELIMINARIES

Let $L^1 = L^1(J)$ be the class of Lebesgue integrable functions on $J = [0, T]$ with the standard norm and let $C = C(J)$ be the space of all real-valued functions defined and continuous on J with the standard supremum norm.

This section collects some definitions and results needed in our further investigations.

Assume that the function $f(t,x) = f: (0,1) \times R \to R$ satisfies Carathèodory conditions, *i.e.*, measurable in t for any $x \in R$ and continuous in x for almost all $t \in J$. Then to every function $x(t)$ being measurable on the interval J we may assign the function

$$(Fx)(t) = f(t, x(t)),\ t \in (0,1),$$

the operator F defined in such a way, is called the superposition (or Nemytskii) operator with the generating function f.

This operator is one of the simplest and most important operators investigated in the nonlinear functional analysis and in the theories of differential, integral and functional equations (see [32], [3-7] and [33]).

Furthermore, for every $f \in L^1$ and every $\psi: J \to J$, we define the superposition operator generated by the functions f and ψ

CHAPTER 12

Non-linear Set-Valued Delay Functional Integral Equations of Volterra-Stieltjes Type: Existence of Solutions, Continuous Dependence and Applications

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Abstract: In this chapter, we established two existence theorems for the non-linear Volterra-Stieltjes integral inclusion. The continuous dependence of the solutions on the delay functions, g_i $(i = 1,2)$ and on the set of selections, will be proved. The non-linear Chandrasekhar set-valued functional integral equation and a non-linear Chandrasekhar quadratic functional integral equation, also the set-valued fractional orders integral equation, are studied as an application. An initial value problem of fractional-orders set-valued integro-differential equation will be considered.

Keywords: Non-linear functional integral equation, Volterra-Stieltjes integral inclusion, Chandrasekhar quadratic integral equation, Function of bounded variation, Continuous dependence, Differential inclusion, Delay function.

1. INTRODUCTION

The non-linear Volterra-Stieltjes type integral operator

$$Tx(t) = \int_0^t f(s, x(s))\, d_s g(t, s), \; t \in I = [0, T]$$

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Praveen Agarwal & Shilpi Jain (Eds.)

has been studied, recently by J.Banas (see [1,2]), and has been studied by some authors(see [3, 4]), and references therein.

Here we discuss with the non-linear set-valued delay functional integral equations of Volterra-Stieltjes type

$$x(t) \in a(t) + \int_0^{\varphi(t)} F_1(t,s,x(s), \int_0^{\varphi(s)} f_2\big(s,\theta,x(\theta)\big) d_\theta g_2(s,\theta). \quad \textbf{(1)}$$

The existence of solutions in the class of continuous functions $x \in C[0,T]$ and the continuous dependence on the delay function φ and the set of selections, of the set-valued functions F, S_F are proved.

As applications of (1) the non-linear Chandrasekhar set-valued functional integral equation

$$x(t) \in a(t) + \int_0^{\varphi(t)} \frac{t}{t+s} F_1(b_1(s)x(s), \int_0^{\varphi(s)} \frac{s}{s+\theta} b_2(s)x(\theta)d\theta, \quad \textbf{(2)}$$

the delay Chandrasekhar quadratic functional integral equation

$$x(t) = a(t) + \int_0^{\varphi(t)} \frac{t}{t+s} b_1(s)x(s) \cdot (\int_0^{\phi(s)} \frac{s}{s+\theta} b_2(s)x(\theta)\, d\theta)ds, \ t \in I, \quad \textbf{(3)}$$

the set-valued fractional orders integral equation

$$x(t) \in p(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F_1(t,s,x(s), \int_0^s \frac{(s-\theta)^{\beta-1}}{\Gamma(\beta)} f_2(s,\theta,x(\theta))d\theta)ds, \ t,s \in I. \quad \textbf{(4)}$$

and the set-valued fractional-order integro-differential equations

$$\frac{dx(t)}{dt} \in I^\alpha F_1(t,x(t),D^\gamma x(t)), \ t \in (0,T], \quad and \quad x(0) = x_o, \quad \textbf{(5)}$$

where $\alpha, \beta, \gamma \in (0,1)$, will be considered. For properties and applications on differential inclusion (5) (see [5 - 7]) and reference therein.

This paper is organized as follows: In Section 2, we recall some useful preliminaries. In Section 3, we investigate existence results for single-valued problem. In Section 4, we discuss the uniqueness of the solution for the Volterra-Stieltjes integral equation. Also, deals with the existence of continuous dependence of solutions for functional integral equations on delay function and functions $g_i, (i = 1,2)$, and some applications explain. While in Section 5, conditions are added to our problem in order to obtain a new existence theorem with the application. In the last Section 6, we present the existence results for a set-valued problem with continuous dependence on the set S_{F_1}, we discuss some special cases of inclusion by presenting the existence of solutions for the Set-valued Chandrasekhar non-linear quadratic functional integral equation and Set-valued Volterra functional integral equation of fractional order, and, as an application, the set-valued fractional-order integro-differential equations will be considered.

2. PRELIMINARIES

This section is devoted to providing the notation, definitions, and other auxiliary facts that will be needed in our further study.

At the beginning, assume that E is a Banach space with the norm $\|.\|_E$. For an interval, $I = [0, T]$, where $T < \infty$ and denote by $C = C(I, E)$, the space consisting of all continuous functions defined on I and taking values in the space E. This space will be furnished with the sup-norm.

$$\| x \|_C = \sup_{t \in I} |x(t)|.$$

We will accept the following axiomatic definition and theorem of the concept of a set-valued map.

Definition 2.1 *Let F be a set-valued map defined on a Banach space E, f is called a selection of F if $f(x) \in F(x)$, for every $x \in E$ and we denote by*

$$S_F = \{f : f(x) \in F(x), x \in E\},$$

the set of all selections of F (For the properties of the selection of F see [8-10]).

Definition 2.2 *[9] A set-valued map F from $I \times E$ to family of all non-empty closed subsets of E is called Lipschitzian if there exists $k > 0$ such that for all $t \in I$ and all $x_1, x_2 \in E$, we have*

$$h(F(t, x_1), F(s, x_2)) \leq k(|t - s| + |x_1 - x_2|), \quad \textbf{(6)}$$

Where $h(A, B)$ is the Hausdorff distance between the two subsets $A, B \in I \times E$ (properties of the Hausdorff distance (See [11])).

CHAPTER 13

Certain Saigo Fractional Derivatives of Extended Hypergeometric Functions

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Abstract: This article aims to establish Saigo fractional derivatives of extended hypergeometric functions. Some special cases of these integrals are also derived.

Keywords: Beta function, Gamma Function, Saigo-fractional integral operator, Saigo-fractional derivative operator, Reimann-Liouville fractional integral operator, Reimann-Liouville fractional derivative operator, Hadamard product, Gauss hypergeometric function, Confluent hypergeometric function.

1. INTRODUCTION

In recent years, many extensions and generalizations of special functions witnessed a significant evolution. This modification in the theory of special functions offers an analytic foundation for the many problems in mathematical physics and engineering sciences, which have been solved and have various practical applications. The theory of special functions revolves around the two most important basic special functions, *i.e*., the beta function and the Gamma function because most of the special functions are expressed either in terms of the beta function or the Gamma function.

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The classical Euler beta function is defined as follows [1]:

$$B(x_1, x_2) = \int_0^1 t^{x_1-1}(1-t)^{x_2-1}dt, \qquad \Re(x_1), \Re(x_1) > 0. \qquad \textbf{(1)}$$

Gamma function is defined as follows [1]:

$$\Gamma(x_1) = \int_0^\infty t^{x_1-1}\, e^{-t}dt, \qquad \Re(x_1) > 0. \qquad \textbf{(2)}$$

Further, the mathematical and physical applications of hypergeometric functions are found in various areas of applied mathematics, mathematical physics, and engineering.

The Gauss hypergeometric function is a solution of a homogenous second-order differential equation which is called the hypergeometric differential equation, and it is given by

$$z(1-z)\frac{d^2w}{dz^2} + (c - (a+b+1)z)\frac{dw}{dz} - abw = 0. \qquad \textbf{(3)}$$

The Gauss hypergeometric function $_2F_1$ is defined as [2]:

$$_2F_1(a, b, c; z) = F(a, b, c; z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}, \qquad \textbf{(4)}$$

where $(u)_k$ represents the Pochhammer symbol defined below:

$$(u)_k = \frac{\Gamma(u+k)}{\Gamma(u)} = \begin{cases} 1 & k = 0; u \in \mathbb{C}/\{0\}, \\ u(u+1).\,.\,.\,.\,.\,(u+k-1) & k \in \mathbb{N}; u \in \mathbb{C}. \end{cases}$$

Later Kummer replaced the parameter z by $\frac{z}{b}$ d taking limit $b \to \infty$ in the equation (3), then the hypergeometric differential equation becomes a confluent hypergeometric differential equation or Kummer's equation.

$$z\frac{d^2w}{dz^2} + (c - z)\frac{dw}{dz} - aw = 0. \qquad \textbf{(5)}$$

The confluent hypergeometric function is the solution of the above differential equation (5).

A confluent hypergeometric function is defined as [2]:

$$ {}_1F_1(a,c;z)=\Phi(a,c;z)=\sum_{k=0}^{\infty}\frac{(a)_k}{(c)_k}\frac{z^k}{k!}, \quad \textbf{(6)} $$

Very recently, Goyal *et al*. [3] introduced an extension of the beta function using the Wiman function, thus studying various properties and relationships of that function:

$$ B_{(u_1,u_2)}^{(u)}(y_1,y_2)=\int_0^1 t^{y_1-1}(1-t)^{y_2-1}E_{u_1,u_2}\left(-u\big(t(1-t)\big)^{-1}\right)dt, \quad \textbf{(7)} $$

where, $\min\{\Re(y_1),\Re(y_2)\}>0$, $\Re(u_1)>0$, $\Re(u_2)>0$, $u\geq 0$, and $E_{u_1,u_2}(z)$ is a 2-parameter Mittag-Leffler function given by [4].

Motivated by the above work, Jain et al. [5] extended Gauss hypergeometric function, and confluent hypergeometric function by using the above-extended beta function and studied various properties of these extended functions. They also studied the increasing or decreasing nature (monotonicity), log-concavity, and log-convexity of the extended beta function defined in [3].

$$ F_{(s_1,s_2)}^{(s)}(q_0,q_1,q_2;z)=\sum_{k=0}^{\infty}\frac{(q_0)_k B_{(s_1,s_2)}^{(s)}(q_1+k,q_2-q_1)}{B(q_1,q_2-q_1)}\frac{z^k}{k!}, \quad \textbf{(8)} $$

Where, $\Re(q_2)>\Re(q_1)>0$, $\Re(s_1)>0$, $\Re(s_2)>0$, $s\geq 0$, $|z|<1$ and d $B_{(s_1,s_2)}^{(s)}$ (w_1,w_2) is the extended beta function.

The extended confluent hypergeometric function is defined as [5]:

$$ \Phi_{(s_1,s_2)}^{(s)}(q_1,q_2;z)=\sum_{k=0}^{\infty}\frac{B_{(s_1,s_2)}^{(s)}(q_1+k,q_2-q_1)}{B(q_1,q_2-q_1)}\frac{z^k}{k!}, \quad \textbf{(9)} $$

where, $\Re(q_2)>\Re(q_1)>0$, $\Re(s_1)>0$, $\Re(s_2)>0$, $s\geq 0$, and $B_{(s_1,s_2)}^{(s)}(w_1,w_2)$ is the extended beta function.

The concept of Hadamard product(convolution) of the functions f and g is very important for our results. Hadamard product of f and g defined as follows [6]:

CHAPTER 14

Some Erdélyi-Kober Fractional Integrals of the Extended Hypergeometric Functions

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Abstract: This paper aims to establish some new formulas and results related to the Erdélyi-Kober fractional integral operator applied to the extended hypergeometric functions. The results are expressed as the Hadamard product of the extended and confluent hypergeometric functions. Some special cases of our main results are also derived.

Keywords: Gamma Function, Beta function, Erdélyi-Kober fractional integral operators, Hadamard product, Gauss hypergeometric function, Confluent hypergeometric function, extended hypergeometric functions.

1. INTRODUCTION AND PRELIMINARIES

In the last few years, many generalizations of special functions with different kernels witnessed a significant evolution. This modification in the theory of special functions offers an analytic foundation for the many scientific problems in mathematical physics, biology, and engineering sciences, which have been solved and have many practical uses.

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Beta and Gamma functions are two important members of a class of special functions which play a vital role in the theory of special functions. Many special functions are expressed either in terms of the beta function or the Gamma function. The gamma function is defined as follows [1].

$$\Gamma(y_1) = \int_0^\infty t^{y_1-1}e^{-t}dt, \Re(y_1) > 0. \quad \textbf{(1.1)}$$

The classical Euler beta function is defined as follows [1].

$$B(y_1, y_2) = \int_0^1 t^{y_1-1}(1-t)^{y_2-1}dt, \Re(y_1), \Re(y_2) > 0. \quad \textbf{(1.2)}$$

Further, the mathematical and physical uses of hypergeometric functions are found in various areas of applied mathematics, mathematical physics, and engineering. The Gauss hypergeometric function is a solution of a homogenous second-order differential equation which is called the hypergeometric differential equation, and it is given by:

$$z(1-z)\frac{d^2w}{dz^2} + (c-(a+b+1)z)\frac{dw}{dz} - abw = 0 \quad \textbf{(1.3)}$$

The Gauss hypergeometric function ${}_2F_1$ is defined as [2]:

$${}_2F_1(a, b, c; z) = F(a, b, c; z) = \sum_{k=0}^\infty \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}, \quad \textbf{(1.4)}$$

where $(u)_k$ represents the Pochhammer symbol defined below:

$$(u)_k := \frac{\Gamma(u+k)}{\Gamma(u)} = \begin{cases} 1 & k = 0; u \in \mathbb{C}\backslash\{0\}, \\ u(u+1)\cdots(u+k-1) & k \in \mathbb{N}; u \in \mathbb{C}. \end{cases}$$

Series representation and integral representation of Gauss hypergeometric function ${}_2F_1$ is defined as [2]:

$$F(r_0, r_1, r_2; z) = \sum_{n=0}^\infty \frac{B(r_1+n, r_2-r_1)}{B(r_1, r_2-r_1)}(r_0)_n\frac{z^n}{n!}, \quad \textbf{(1.5)}$$

where $\Re(r_2) > \Re(r_1) > 0$ and $|z| < 1$.

$$F(r_0, r_1, r_2; z) = \frac{1}{B(r_1, r_2-r_1)}\int_0^1 t^{r_1-1}(1-t)^{r_2-r_1-1}(1-zt)^{-r_0}dt. \quad \textbf{(1.6)}$$

Then later, Kummer changes the parameter z by $\frac{z}{b}$ and taking limit $b \to \infty$ in the (1.3), then the hypergeometric differential equation becomes a confluent hypergeometric differential equation or Kummer's equation.

$$z\frac{d^2w}{dz^2} + (c-z)\frac{dw}{dz} - aw = 0 \qquad \textbf{(1.7)}$$

The confluent hypergeometric function is the solution of the above differential equation (1.7). A confluent hypergeometric function is defined as [2]:

$${}_1F_1(a,c;z) = \Phi(a,c;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k}\frac{z^k}{k!}. \qquad \textbf{(1.8)}$$

Series representation and integral representation of a confluent hypergeometric function are defined as [2]:

$$\Phi(r_1,r_2;z) = \sum_{n=0}^{\infty} \frac{B(r_1+n,r_2-r_1)}{B(r_1,r_2-r_1)}\frac{z^n}{n!}, \qquad \textbf{(1.9)}$$

where $\Re(r_2) > \Re(r_1) > 0$;

$$\Phi(r_1,r_2;z) = \frac{1}{B(r_1,r_2-r_1)}\int_0^1 t^{r_1-1}(1-t)^{r_2-r_1-1}e^{zt}dt. \qquad \textbf{(1.10)}$$

Recently, Goyal *et al.* [3] studied a new extension of the beta function using the 2-parameter Mittag-Leffler function as a kernel and derived some important results for extended beta functions.

$$B_{(a_1,a_2)}^{(a)}(x_1,x_2) = \int_0^1 t^{x_1-1}(1-t)^{x_2-1}E_{a_1,a_2}(-a(t(1-t))^{-1})dt, \qquad \textbf{(1.11)}$$

where $\min\{\Re(x_1),\Re(x_2)\} > 0$, $\Re(a_1) > 0, \Re(a_2) > 0$, $a \geq 0$ and $E_{a_1,a_2}(z)$ is a 2-parameter Mittag-Leffler function given by [4]. Then later, Jain et al. [5] introduced new extensions of the Gauss hypergeometric function and confluent hypergeometric function by using the above-extended beta function and studied various properties of these extended functions. They also have inequalities of the extended beta function defined in [3]. The extended confluent hypergeometric function is defined as [5]:

$$\Phi_{(a_1,a_2)}^{(a)}(b_1,b_2;z) = \sum_{r=0}^{\infty} \frac{B_{(a_1,a_2)}^{(a)}(b_1+r,b_2-b_1)}{B(b_1,b_2-b_1)}\frac{z^r}{r!}, \qquad \textbf{(1.12)}$$

CHAPTER 15

On Solutions of the Kinetic Model by Sumudu Transform

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Abstract: This paper investigates the kinetic model with four different fractional derivatives. We obtain the solutions of the models by Sumudu transform and demonstrate our results with some figures. We prove the accuracy of the Sumudu transform by some theoretical results and applications.

Keywords: Sumudu transforms, Fractional derivatives, Kinetic model.

1. INTRODUCTION

Mathematical modeling formulas have been used to predict the growth of microorganisms, the spread of epidemics, and drying kinetic models are some real-world problems. Modeling of mass transfers is created with Fick's law. For the fractional version of Fick's law, see [1] and the references in this work. Although the classical Lewis model defines the exponential behavior of diffusion, the fractional Lewis model can also use the non-exponential behavior of diffusion [2]. The speed of the grain drying operation varies with the supply, such as temperature, airflow and mechanical drying systems. Additionally, drying kinetic models express the time required for optimum moisture loss. We consider [3]:

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Praveen Agarwal & Shilpi Jain (Eds.)

$$\frac{dN}{dt} = D\nabla^2 N, \tag{1.1}$$

Here, D is the diffusion coefficient, and N is the moisture content of the food. We present the Lewis model [4] as:

$$\frac{dN}{dt} = -k(N(t) - N_e), \tag{1.2}$$

where N_e is equilibrium moisture content, and k is a constant of drying has dimension min^{-1}. Then, we get:

$$N(t) = (N_0 - N_e)\exp(-kt) + N_e, \tag{1.3}$$

Here, N_0 is the initial moisture content inside the food.

Fractional analysis has gained a lot of attention recently. The most crucial reason for this is that it has implementations in real-world problems and gives better results in comparison. Fractional calculus has also been used to model physical and engineering processes. Recently, fractional calculus has played a huge role in various fields, such as mechanics, electricity, chemistry, biology, and economics, especially in control theory. Atangana *et al.* [5] studied the generalized mass transport equation. The analysis of new trends in the fractional differential equation has been investigated in [6, 7]. Abdeljawad *et al.* [8] surveyed fractional differences and integration by parts. Some classes of ordinary differential equations have been worked on by Jarad *et al.* [9]. Afshari *et al.* [10] examined a new fixed point theorem with an implementation of a coupled fractional differential equations system. KÃrt *et al.* [11] studied a certain bivariate Mittag-Leffler function in 2020. Integral transforms and some special functions were investigated by Saxena *et al.* [12] and Fernandez *et al.* [13] also conducted various studies on this subject. The purpose of this study, because of the usefulness and significance of the fractional differential equations in certain physical problems, is to give numerical results using the Sumudu transformation for the kinetic model (1.2).

2. MATHEMATICAL BACKGROUND

The Sumudu transformation was first introduced by Watagula [14]. Weerakoon [15] investigated the complex inverse formula for Sumudu transform. Asiru has studied the implementation of the Sumudu transform [16], discrete dynamical system equations [17] and more [18]. Sumudu transform is a theoretical dual of the Laplace transform. Sumudu transforms to convolution-type integral equation was applied by Belgacem *et al.* [19]. Additionally, Belgacem *et al.* [20] mentioned many features of the Sumudu transform. For more details, we refer readers to [21-26]. Purohit *et al.* [27] examined an application of Sumudu transform for a fractional kinetic equation. Nisar *et al.* [28-30] also studied these types of equations.

Definition 2.1 *Over the set of functions,*

$$A = \{g(t) | \exists N, \tau_1, \tau_2 > 0, |g(t)| < N\exp(|t|/\tau_j, \text{ if } t \in (-1)^j \times [0, \infty)\}, \quad \textbf{(2.1)}$$

the Sumudu transform is defined as [20]:

$$G(u) = S[g(t)] = \int_0^\infty f(ut)\exp(-t)dt, \quad u \in (-\tau_1, \tau_2). \quad \textbf{(2.2)}$$

The Sumudu transform changes domain size and shape, not units, unlike Laplace transform. ODEs solved by the Laplace transform can be solved by Sumudu transform and almost vice versa, except for some possibly artificially manufactured examples. The primal thing is that Sumudu is more natural and easier to understand, so there are lots of Sumudu transform's applications in the literature. Actually, this transform is linear, protects linear functions, and hence especially does not change units [14].

Definition 2.2 *Let* $\eta, \zeta: [0, \infty) \to \Re$, *then the convolution of* η, ζ *is [31]*

$$(\eta * \zeta) = \int_0^t \eta(t-u)\zeta(u)du \quad \textbf{(2.3)}$$

and assume that $\eta, \zeta: [0, \infty) \to \Re$, then we have

$$S\{(\eta * \zeta)(t)\} = uS\{\eta(t)\}S\{\zeta(t)\}. \quad \textbf{(2.4)}$$

Definition 2.3 *The Caputo fractional derivative is presented as follows [32];*

$${}_a^C D^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-z)^{n-\alpha-1} g^{(n)}(z)dz \quad \textbf{(2.5)}$$

where $\alpha \in \mathbb{C}, Re(\alpha) > 0, n = [Re(\alpha)] + 1$.

Lemma 2.4 *The Sumudu transform of Caputo fractional derivative is defined by [31]*

$$S[{}_0^C D_t^\alpha g(t)] = \frac{G[u]-g(0)}{u^\alpha}, \quad \textbf{(2.6)}$$

SUBJECT INDEX

A

Adaptive 3, 4, 13, 24
anti-synchronization 24
control methods 3, 4
synchronization 4, 13
Algorithm 74, 144
efficient 74
Analysis of fractional order dynamical systems 2
Analytic 113, 244, 259
foundation 244, 259
mapping 113
Anti-periodic BVP 173
Anti-synchronization 1, 3, 4, 6, 7, 9, 23, 24, 33
behavior 23
error 9
Arzela-Ascoli theorem 204, 226, 234
Ascoli’s theorem 180, 194
Atangana-Baleanu 273, 279
derivatives 279
fractional 273
Autonomous 1, 9, 10, 11, 23
chaotic systems 1, 9, 10, 11, 23
system 11, 23

B

Baleanu derivatives 157
Banach contraction theorem 159
Bernstein 73, 74, 75, 77, 83, 84, 85, 86, 87, 88, 89
Bézier techniques 74
polynomials (BP) 73, 74, 75, 77, 83, 84, 85, 86, 87, 88, 89
Bessel functions 98, 100
Bezier coefficients 74
Bohmann-Korovkin-type approximation theorem 74
Bohnentblust-Karlin theorem 170
Boundary conditions 118, 146, 160, 174, 175
anti-periodic 174
periodic 160
Bounded variation 114, 222
Brownian motion 186

C

Caputo 1, 2, 5, 136, 137, 138, 139, 140, 141, 143, 154, 157, 158, 273, 274, 275, 279, 280
definition 138, 139
derivatives 2
Fabrizio and Atangana-Baleanu derivatives 279
Fabrizio fractional derivative 273, 275
fractional 136, 138
implementing 136
sense 1
time-fractional 136, 137, 138, 139, 140, 141, 143
Carathèodory 185, 187, 191, 196, 199, 206
conditions 187, 191, 199, 206
theorem 185, 191
Cardiac tissues 2
CFC fractional 273
Chaotic dynamical 2
system mimics 2
systems 2
Chaotic systems 2, 3, 4, 6, 33
anti-synchronize 4
integer-order 33
synchronizing 2
Compactness 159, 168, 170
condition 159, 170
weakening 159
Computations, numerical 154
Conditions 58, 59, 64, 116, 118, 159, 161, 196, 211, 220, 222, 235, 236
growth 118, 196
Confluent hypergeometric function 244, 245, 246, 254, 255, 259, 261, 265, 266
Conformal mapping 112

Constant proportional caputo (CPC) 273, 274, 277, 280
Control theory, nonlinear 4
Convergence 74, 79, 112, 113, 195
analysis 79
theorem 74, 112, 113
dominated 195
Convex 158, 166, 171
nonempty 158
subsets 166, 171

D

Darbo's fixed point theorem 169
Derivative(s) 1, 4, 13, 24, 73, 122, 124, 126, 129, 138, 145, 244, 247, 248, 252, 262
fractional order time 1, 4
integer-order 262
operators 122, 124, 126, 129, 138, 244, 247, 248, 252, 262
orders 145
Differential equations 39, 43, 45, 47, 48, 49, 99, 100, 157, 158, 185, 245, 260, 261, 271
system 271
Differential operator 2, 5
Diffusion 146, 270, 271
coefficient 271
Diffusive processes 189
Dirac-delta function 42
Dirichlet 112, 116, 138
boundary conditions 138
series 112, 116
Drive system 6, 9, 11, 13, 16, 19, 22, 23, 24, 27, 30, 33
Drying kinetic 274, 275, 276, 277
Dynamical 2, 33
behavior 2
systems 2
nonlinear 2, 33
Dynamics 1, 2, 247, 262
fluid 247, 262
nonlinear 2

E

Efficiency 73, 74, 83, 136, 145, 280
computational 73
Electric circuit analysis 39
Electrodynamics 213
Equation 8, 23, 125, 126, 129, 131, 138, 139, 140, 141, 142, 143, 144, 145, 146, 158, 187, 213, 214, 232, 233, 251, 264, 271, 272, 274, 277
algebraic 233
analytical solution of 146
dimensional linear diffusion 145
discrete dynamical system 271
fractional kinetic 272
hybrid differential 187, 213
linear 140
perturbed differential 158
Erdelyi-Kober 186, 188, 189, 190
operators 186, 188, 189
sense of 186, 189
Erdélyi-Kober fractional 189, 265, 266
diffusion 189
integrals of gauss hypergeometric function 265, 266
Expansion 64, 75, 76, 83, 89
asymptotic 64
binomial 75, 76

F

Fick's law 270
Fixed point theorem 158, 169
celebrated 158
for monotone operators 169
Fluid mechanics 2
Formula, mathematical modeling 270
Fourier inversion formula 113
Fractional 1, 2, 5, 39, 40, 41, 43, 47, 48, 49, 137, 138, 145, 146, 157, 158, 189, 190, 239, 240, 247, 252, 262, 265 270, 271, 274
boundary conditions 145, 146
calculus operators 247, 262

coupled 271
derivatives 1, 5, 41, 262, 270, 274
exponents 39, 262
integral formulas 252, 265
linear 39
partial differential equations (FPDEs) 39, 40, 49, 137, 145
Fractional integration 186, 188
operation 188
operator 186
Fractional order 1, 2, 4, 5, 9, 10, 11, 187, 191, 196, 197, 198, 205, 206, 214, 219, 220, 239, 240, 241
derivatives 4
dynamical systems 2
functional quadratic equation of 197
integral inclusions of 240
quadratic integral equation of 191, 196, 197, 198, 214
set-valued 219, 220, 241
Functional analysis 187, 262
nonlinear 187
Function projective synchronization 3

G

Galerkin and collocation methods 74
Gauss 244, 245, 248, 250, 254, 259, 260, 265, 266
hypergeometric function 244, 245, 248, 250, 254, 259, 260, 265, 266
Gauss-Seidel 136, 137
iteration 137
methods 136
General fractional order 187, 189, 191, 193, 195, 197, 199, 201, 203, 205, 207, 209, 211, 213
Generalized hypergeometric function 64, 94, 98, 251, 252, 253, 254, 255, 256
Genocchi 121, 122, 123, 124, 126, 128, 130, 132, 134
numbers and Genocchi polynomials 123
polynomials 121, 122, 123, 124, 126, 128, 130, 132, 134
Gould-Hopper 122, 123, 128
Genocchi polynomials (GHGP) 128
polynomials (GHP) 122, 123
Grain drying 270, 280
operation 270
process 280

H

Half-sweep preconditioned gauss-Seidel 143
Hausdorff 168, 169, 221
distance 221
measure of noncompactness 168, 169
HSPGS method 137, 143, 144, 145, 154
Hybrid functional 212
Hypergeometric functions 62, 63, 64, 65, 67, 69, 71, 245, 247, 249, 251, 253, 254, 255, 260, 261
derivative gauss 254
Hypergeometric series 62, 100

I

Image processing 247, 262
Integral equation 185, 186, 196, 197, 198, 199, 205, 206, 207, 211, 212, 219, 220, 223, 224, 236, 240, 241
set-valued functional 219, 220, 240, 241
single-valued 236
Iterative methods 136, 137, 144, 145, 147, 154

K

Kakutani theorem 167, 172
Koebe function 95
maps 95
Kummer’s equation 245, 261

L

Laplace transform method 113
Lax equivalence theorem 141
Linear 42, 78, 170, 214

change 42
combinatory 78
functional argument 214
operator 170
Linear system 136, 140, 143, 154
preconditioned 143
Lipschitz condition 211, 227, 236
Lommel functions 98, 99
Lorenz system 11
Lyapunov 4, 8, 9
function 8
stability method 4
stability theory 4, 9

M

Mittag-Leffler function 43, 53, 274
Monotone 161, 182
mapping 161
single/mutivalued operators 182
Monotone self 161, 162
mapping 161
operators 162
Monotonicity 202, 246
Moulton method 4, 33

N

Neutron transport 185
Nonlinear 7, 214, 219
control function 7
functional arguments 214
Non-linear functional integral equation 219

O

Operational techniques 121

P

PDEs, nonlinear 137
Perron's condition 211
Picard and Adomian methods 198
Plancherel theorem 118
Pochhammer symbol 54, 245, 260
Properties, fundamental 74

Q

Quadratic integral equations (QIEs) 185, 214

R

Riemann-Liouville fractional 55, 58, 138, 190, 252
calculus 58
integration 190

S

Stability analysis 141
Stieltje's integrability 222
Stochastic processes 189
Struve functions 98
Sumudu transformation 271, 277
Sumudu transform 273
of ABC fractional 273
of CFC fractional 273
Synchronization 2, 3, 4, 11, 13
behavior 11
error vectors 13
method 4
of chaos 2
of chaotic dynamical systems 2
of chaotic systems 3, 4
Systems 1, 2, 3, 4, 6, 7, 9, 10, 11, 13, 24, 74, 247, 262, 270
chaotic communication 2
drive and response 3, 13, 24
hyperchaotic 4
mechanical drying 270

T

TBS technique 90
Time-fractional orders 150, 151, 152, 153
Transcendental functions 73

V

Viscoelasticity 2
Visual mathematical analysis 133
Volterra 240
  non-linear 240

W

Weierstrass theorem 80
Whittaker transform 69
Wiman function 246

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